Nonextensivity, Complexity and Nonlinearity in Space Plasmas

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Abstract Experimental time series, extracted from many and different space plasma systems corresponding to, solar wind, magnetospheric and other space plasma systems reveal common dynamical, geometrical, or statistical characteristics. Such characteristics are the low dimensionality, the typical intermittent turbulence multifractality, the temporal or spatial multiscale correlations and power laws scale invariance, non Gaoussianity and others. This universal aspect of experimental time series profiles was understood in the past as the chaos or SOC universality. However, after two or three decades of theoretical development in understanding of the nonlinearity and complexity, we can give a more compact theoretical description of the underline universal physical processes that produce the experimental time series complexity. Finally, in this study, we present and explain the modern complex set of theoretical concepts from the point of view of physics as the unification theory of nonlinear theory of non-equilibrium plasma systems as well as the presupposed theoretical framework of time series analysis of space plasma charachteristics.

Keywords: Nonextensivity, multifractality, intermittent turbulence, fractional plasma theory, space plasma percolation.

Introduction

Signatures of low dimensional chaos in magnetospheric and solar plasmas were discussed in a series of papers of Pavlos et al., Athanasiou et al., Karakatsanis and Pavlos (Pavlos et al., 1992a,b,c, 1994, 1999, 2003, 2004; Athanasiou, 2001, 2003; Karakatsanis and Pavlos, 2008),while the nonequilibrium phase transition or topological phase transition between high dimensional SOC and low dimensional Chaos was indicated in a series of studies (Pavlos et al., 2011, 2012a,b,c, 2014, 2015, 2016; Karakatsanis et al., 2012, 2013). In previous studies, the hypothesis of Tsallis non-extensive statistics in magnetosphere, sunspot dynamics, solar flares, solar wind and space plasma in general was tested and verified. Also, as we have shown in these studies, Tsallis q-entropy principle is the deep core of space plasma self-organization, producing low dimensionality, intermittent turbulence, non-Gaussian, long range correlations, anomalous diffusion, fractality and other characteristics of complexity. Some high lights of complex character of space plasmas are summarized below.

Especcialy, we can support the concept that the older reductionistic universality of particles and fields-forces in plasma dynamics is extended to the modern universality of multi-scale complex processes from the microscopic to the macroscopic level of different plasma systems. Furthermore, we can claim the existence of a basic and universal organizing principle creating complex spatiotemporal and multiscale different physical structures or different dynamical scenarios at every physical scale level of a plasma system. The best physical representation of the underline universal organizing principle is the well-known entropy principle. After the appearance of Tsallisnonextensive q-extension of the Boltzmann-Gibbs (BG) statistics and the introduction of Tsallis q-entropy (Sq) (Tsallis, 1988), one can understand in a deep level the universality of entropy principle as the physical manifestation of the universal and multi scale organizing principle. That is the physical multiscale organizing principle is identical to the entropy principle, according to which nature creates complex structures everywhere, from a microscopic to macroscopic level, trying to succeed the extremization of the TsallisSq entropy that includes the well-known Boltzmann-entropy as the $q = 1$ case.

The Sq entropy principle is harmonized with the *q* extension of the classic and Gaussian central limit theorem (*q*-CLT). The q-extension of CLT corresponds to the Levy a-stable extension of the Gaussian attractor of the classic statistical theory. The *q*-CLT is related to the Tsallis *q*-triplet theory of random time series with non-Gaussian statistical profile. During the last years, we have tested the universal character of Tsallis q-triplet applied to different experimental time series obtained from terrestrial magnetospheric as well as extraterrestrial space plasma systems. Moreover, Tsallis q-extended entropy principle can be used to unify internally some new dynamical manifestations of plasma systems such as the spatiotemporal fractional dynamics, anomalous diffusion processes and the strange dynamics of Hamiltonian and dissipative dynamical systems, the intermittent turbulence theory, the fractional topological and percolation phase transition processes, according to Zelenyi and Milovanov non equilibrium and nonstationary states (NESS) theory (Zelenyi and Milovanov, 2004), as well as to unify the non-equilibrium Renormalization Group Theory (RGT) of distributed dynamics and the physical process of reduction of dynamical degrees of freedom. Finally, in this study and from the physical point of view, we present and explain the above mentioned set of theoretical concepts as the unification theory of nonlinear theory of non-equilibrium plasma systems as well as the presupposed theory of space plasmas time series analysis. Moreover, the nonlinearity in space plasma dynamics can generate intermittent turbulence with the typical characteristics of the anomalous diffusion process and strange topologies of stochastic space plasma fields (the velocity and magnetic fields) caused by the strange dynamics and strange kinetics (Zaslavsky, 2002). In addition, according to Zelenyi and Milovanov (2004), the complex character of the space plasma system includes the existence of nonequilibrium (quasi)-stationary states (NESS), having the topology of a percolating fractal set. The stabilization of a system near the NESS is perceived as a transition into a stationary turbulent state determined by self-organization processes according to the principle of Sq entropy extramization. The extramization of Sq entropy createslong-range correlation effects,which manifest themselves as a strange non-Gaussian behavior of kinetic processes near the NESS plasma state. More analytically we have the following characteristics:

(a) Far from equilibrium, the development of spatiotemporal plasma structures, including long-range correlations according to the q-extension of CLT, can be indicated by the estimation of Tsallis q-triplet. The most importat feature is that, far from equilibrium, the q-triplet (q_{sen}, q_{stat}, q_{rel}) can be significantly differrent from the equilibrium Gaussian profile, in which $q_{sen} = q_{stat} = q_{rel} = 1$.

(b) Tsallis non-extensive entropy theory is verified by the multifractal and multiscale strange character of the underlying space plasma phase space.The far from equilibrium strange phase space includes anomalous topology connected to multi-scaling and multifractality. This anomalous character of phase space,is projected to the corresponding multifractal and anomalous topology and anomalous diffusion of the dissipation regions in the physical space. In order to study these characteristics of phase space and physical space we estimate the multifractal spectrum of dimensions $f(a)$ and D_q, according to the multifractal theory, as well as the structure function (S_p) and its scaling exponent spectrum [$J(p)$], which can be different from Kolmogorov's first theory (K41) predictions. According to Frisch's predictions for non-Gaussian multifractal and intermittent dissipation processes (Frisch, 1995), the scaling exponent spectrum $J(p)$ satisfies the relation:

$$\frac{dJ(p)}{dp} = h_*(p) \neq 0 \qquad (1)$$

where $h_*(p)$ characterizes the fractal dimension $D(h)$ of the fractal dissipation region by the relation:

$$\frac{dD}{dp} \equiv D'(h(p)) = p \qquad (2)$$

The fractal dimension $D(h)$ and the exponents of the structure function $J(p)$ are related to a Legendre transformation:

$$J(p) = \inf_{(h)} \left[ph + 3 - D(h) \right] \qquad (3)$$

This relation indicates the fact that, when the dissipation region is multi-fractal, then $D(h) \neq 3$ and $dJ(p)/dp \neq p$. We can notice here that according to the theory of Kolmogorov (K41), the dissipation region is mono-fractal.

(c) For the estimation of correlation dimension D we employ the theories of Takens (Takens, 1981),Grassberger and Procaccia (Grassberg and Procacia, 1983) and Theiler (Theiler, 1991).

(d) The Lyapunov exponent spectrum (λ_i) is obtained by the evolution of small perturbations of the dynamical orbit in the reconstructed state space (Sano and Sawada, 1985).

(e) The color noise pseudo chaos profile is found from the low dimensional chaotic data, exploiting the Theiler method of surrogate data (Theiler et al., 1992).

According to the above theoretical concepts and the analytic description of section (2) of the first part (Pavlos et al., 2012b), below we present the plan for the experimental data analysis of the solar plasma time series:

- By using the singular value decomposition (SVD) method, we can succeed the discrimination of deterministic and noisy (stochastic) components included in the observed signals, as well as the discrimination of discrete dynamical components. Also, we can estimate geometrical and dynamical characteristics of the phase space dynamics.
- Estimation of the Flatness coefficient F for testing Gaussianity – nonGaussianity profile.
- Estimation of the q-stationary (q_{stat}) index of the q-Gaussian through linear correlation fitting of $\ln_q p(z)$ versus z^2, where $\ln_q p(z)$ is the *q*-logarithmic function of the probability distribution function (PDF) $p(z)$, as $z = z_{n+1} - z_n$, ($n = 1,\ldots,N$) corresponds to the first difference of the experimental solar flares time series data (Ferri et al., 2010).
- Determination of the q-relaxation (q_{rel}) index of the q-statistics, according to the relation: $\frac{d\Omega}{dt} = -\frac{1}{T_{q_{rel}}}\Omega^{q_{rel}}$, where Ω denotes the autocorrelation functions or the mutual information function of the experimental time series (Tsallis, 1988, 2009).
- Determination of the q-sensitivity (q_{sen}) index according to the equation: $\frac{1}{q_{sen}} = \frac{1}{a_{min}} - \frac{1}{a_{max}}$, where a_{min} and a_{max} denote zero points of the multifractal exponent spectrum. Theq_{sen}index corresponds to the Kolmogorov-

signal entropy production and Pesin theory (Marsch and Tu, 1997; Lyra and Tsallis, 1998).

- Determination a) structure functions $S(p)=\left\langle |\delta u|^{p}\right\rangle$, where δu is the spatial variation of the bulk plasma flow velocity or other signals and b) the scaling exponent spectrum, according to the relation $S(p)\sim l^{J(p)}$, where l is the length scale of the dissipation (structure function) process (Pavlos et al., 2012c).
- Determination the correlation dimension (D) by using the saturation value of the slopes (D_m) of the correlation integrals (C_m)
- Determination the Lyapunov exponent's spectrum.
- Determination the significance (σ) of the discriminating statistics by using the surrogate method of Theiler (Theiler et al., 1992). For σ-values >3, the null hypothesis can be rejected with the confidence exceeding 99%.

The phenomenology of space plasma complexity, as it was summarized previously, clearly reveals the existence of a dynamical non-equilibrium phase transition process concerning the space plasma dynamics, as the external parameters are changing. This non-equilibrium phase transition process includes the transition between the quiet state to states, which include enhancement of self-organization and intermittency. The results of many previously referenced studies show that the space plasma processes certainly exhibit internal dynamical effects related to the observed transformation of the non-extensivity character and the enhancement of the self-organization process. The previously discussed characteristics of space plasmas are also in agreement with the general theory of complex plasma dynamics. According to Zelenyi and Milovanov (Zelenyi and Milovanov, 2004), the complex character of the solar plasma can be described as non-equilibrium (quasi)- stationary states (NESS) having the topology of a percolating fractal set. These scales include multiscale interactions between fields and particles (currents) and can be related to the simultaneous development of numerous space plasmasinstabilities interfering with each other.

The space plasmas complex states corresponds to the stabilization near the turbulent NESS identified by the generalized symmetries of a fractal disk diffeomorphism to a fractal set at the percolation threshold (Zelenyi and Milovanov, 2004). The structural stability of the NESS as a symmetric turbulent phase maintains due to multi-scale correlations, creating the existence of local extremes in the free energy. In addition, the interpretation of the results indicate the possibility for the existence of solar wind plasma system phase transitions from a weak NESS to a strong NESS as the outcome of cluster interaction in the magnetosphere or the interplanetary space, as well as at the coronal source of the solar wind. Below, we discuss various important experimental and theoretical aspects of space plasma complexity.

Nonextensive Statistical Mechanics

Tsallis, inspired by multi-fractal analysis (Tsallis, 1988), supposed that the BG entropy

$$S_{BG}=-k\sum p_i \ln p_i = k < \ln(1/p_i) > \quad (4)$$

cannot describe the entire complexity of nonlinear dynamical systems. BG statistical theory presupposes ergodicity of the underlying dynamics in the system phase space. However, the complexity of dynamics is far beyond a simple ergodic complexity and can be described by non-extensive Tsallis statistics based on the extended concept of *q*-entropy:

$$S_q = k\left(1-\sum_{i=1}^{N} p_i^q\right)/\left(q-1\right) = k < \ln_q(1/p_i) > \quad (5)$$

For a continuous state space, we have

$$S_q = k\left[1-\int\left[p(x)\right]^q dx\right]/\left(q-1\right) \quad (6)$$

Therefore, Tsallis*q*-extension of statistical physics opened the way for *q*-extension of thermodynamics and general critical dynamical theory for nonlinear systems that are far from thermodynamic equilibrium. According to the Tsallis*q*-extension of the entropy principle, any stationary random variable can be described as the stationary solution of a generalized fractional diffusion equation. For meta-stable stationary solutions of a stochastic process, the maximum entropy principle of BG statistical theory can faithfully be described by the maximum (extreme) of the Tsallis*q*-entropy function. Extremization of Tsallis*q*-entropy corresponds to the *q*-generalized form of the normal distribution function:

$$p_q(x) = A_q\sqrt{\beta}e_q^{-\beta(x-<x>_q)^2} \quad (7)$$

where $A_q=\sqrt{(q-1)/\pi}\Gamma(1/(q-1))/\Gamma((3-q)/[2/(q-1)])$ for $q > 1$,

and $A_q=\sqrt{(1-q)/\pi}\Gamma((5-3q)/[2(1-q)])/\Gamma((2-q)/(1-q))$ for $q < 1$, , $\Gamma(z)$ being the Riemann function.

2.1 Tsallis q-triplet

The manifestation of non-extensive statistics can be realized in three distinct ways, corresponding to Tsallis *q*-triplet triplet ($q_{sen}, q_{stat}, q_{rel}$) (Tsallis, 2004a). These quantities characterize three physical processes: a) *q*-entropy production (q_{sen}), b) relaxation process (q_{rel}), c) equilibrium fluctuations (q_{stat}). The *q*-triplet triplet values characterize the attractor set of the dynamics in the phase space of the dynamics, and they can change when the dynamics of the system is attracted to another attractor set of the phase space. In the case of equilibrium Gaussian (BG) world, the *q*-triplet of Tsallis simplifies to ($q_{sen} = q_{stat} = q_{rel} = 1$), (Tsallis, 2009).

Higher values of q_{sen} parameter mean strengthen of the entropy production rate. The latter is caused by the development of long-range correlations, which creates non-local connections between parts of the system. Also, higher values of the q_{sen} parameter correspond to lower values of *q*-entropy (S_q-entropy). This is caused by the development of order and self-organization process in the complex system. Lower values of q_{rel} parameter indicate decrease in the relaxation time caused by strengthen of the non-local interactions and long-range correlations.

As we have noticed before, Tsallis non-extensive statistical mechanics includes the *q*-generalization of the classic central limit theorem (CLT) as a *q*-generalization of the Levy –Gnedenko central limit theorem (Umarovet al., 2008) applied for globally correlated random variables. The *q*-generalization of CLT based at the *q*-Fourier transform of a *q*-Gaussian can produce an infinite sequence (q_n) of *q*-parameters in agreement with the function, $Z(s) = (1+s)/(3-s)$ $s \in (-\infty, 3)$ and its inverse $z^{-1}(t), t \in (-1, \infty)$. It can be shown that $z(1/z(s)) = 1/s$ and $z(1/s) = 1/z^{-1}(s)$, as well as if $q_1 = z(q)$ and $q_{-1} = z^{-1}(q)$ it follows that: $z\left(\frac{1}{q_1}\right) = \frac{1}{q}, z\left(\frac{1}{q}\right) = \frac{1}{q-1}$ and $q_{-1} + \frac{1}{q_1} = 2$. The set of all *q*-Gaussians $G_q(\beta, x)$ is denoted by:

$$\Im_q = \left\{ bG_q(\beta, x) : b > 0, \beta > 0 \right\} \qquad (8)$$

For *q*-Gaussians the *q*-Fourier transform holds as follows:

$$F_q\left[G_q(\beta; x)\right](\xi) = e_{q_1}^{-\beta_*(q)\xi^2} \qquad (9)$$

where $q_1 = z(q)$, $q < 3$.

The *q*-Fourier transform is defined by Umarov et al. (2008) by the formula:

$$F_q[f](\xi) = \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)dx \qquad (10)$$

For the inverse *q*-Fourier transform we have the following formula:

$$F_{q_{-1}}\left[G_{q_{-1}}(\beta; x)\right](\xi) = e_q^{-\beta_*(q_{-1})\xi^2} \qquad (11)$$

where $q_{-1} = z^{-1}(q)$, $q > -1$ and $\beta_*(s) = \frac{3-s}{8\beta^{2-s} C_s^{2(s-1)}}$

The above extension of Fourier Transform to *q*-Fourier Transform (*q*-FT) permits the extension of statistical independence of the random variables x,y to their *q*-independence according to the relations:

$$F_q(x+y)(\xi) = F_q[x](\xi) \otimes_q F_q[y](\xi) \qquad (12)$$

$$F_{q-1}(x+y)(\xi) = F_{q-1}[x](\xi) \otimes_q F_q[y](\xi) \qquad (13)$$

where $q = z(q_{-1})$. The *q*-independence means independence for $q = 1$ but strong correlation for $q \neq 1$ (Tsallis 2004c, 2009).

The *q*-FT includes a sequence of *q*-parameters $\{\ldots, q_{-2}, q_{-1}, q_0, q_1, q_2, \ldots\}$ in accordance with the relations:

$$\left.\begin{aligned} q_k &= Z_k(q) = Z(Z_{k-1}(q)), \quad k = 1,2,\ldots \\ q_{-k} &= Z_{-k}(q) = Z^{-1}(Z_{1-k}(q)), \quad k = -1,-2,\ldots \\ q_k &= \frac{2q + k(1-q)}{2 + k(1-q)} \quad k = 0, \pm1, \pm2, \ldots \end{aligned}\right\} \qquad (14)$$

The essence of the *q*-CLT (*q*-extension of CLT) concerns the possibility of *q*-generalization of the standard CLT by allowing the random variables that, being summed, correlate through an attractor as follows. If there is the sequence of $x_1, x_2, \ldots, x_N$ of q_k $(k \in Z)$ independent and identically distributed random variables with a finite *q*-mean and a finite second *q*-moment, then their sum $Z_N = x_1 + x_2 + \ldots + x_N$ is *q*-convergent to the attractor of a q_{k-1} Gaussian distribution $Gq_{k-1}(\beta; x)$. The *q*-FT of the attractor $Gq_{k-1}(\beta; x)$ is the distribution $Gq_k(\beta_*; x)$ (*q*-Gaussian). Therefore, the correlations are introduced through the q_k products of q_k - Fourier Transforms.

The *q*-CLT includes three physically significant *q*-parameters known as the *q*-triplet: (P_{att}, P_{cor}, P_{scl}) identified by the identical equations:

$$(P_{att}, P_{cor}, P_{scl}) \equiv (q_{k-1}, q_k, q_{k+1}) \equiv (q_{stat}, q_{rel}, q_{sen}) \qquad (15)$$

The parameter $P_{att} \equiv P_{k-1} \equiv q_{stat}$, describes the statistical attractor $Gq_{k-1}(\beta; x)$ of the *q*-independent random variables. The $P_{cor} \equiv P_k \equiv q_{rel}$, describes the *q*-correlated random variables of the system and the relaxation process of fluctuations toward the attracting stationary state. The $P_{scl} \equiv P_{k+1} \equiv q_{sen}$, describes the scale invariance of the multifractal structure of the system according to the asymptotically scaling form:

$$N^D P_x(x) \sim G\left(\frac{x}{N^D}\right) \qquad (16)$$

where $P_x(x)$ is the probability function of the self-similar attractor (*q*-Gaussian) and *D* is the scaling exponent characterizing the anomalous diffusion process (Baldovin and Stella, 2007):

$$< x^2 > \sim x^{2D} = x^{\delta/2} \qquad (17)$$

where $\boldsymbol{\delta} = \boldsymbol{q}_{k+1} = \boldsymbol{q}_{sen}$. For non-Gaussian dynamics ($\boldsymbol{q}_{k+1} \neq 1$) the statistical attractor of the system dynamics creates multiscale correlations and multifractal structures in the phase space as well as in the physical space through the q-entropy extramization. From this point of view, the $\boldsymbol{q}_{sens}$ parameter describes the non-ergodic q-entropy production of the multiscale correlated process as shifts of the system to the state of the q-Gaussian attractor, where the q-entropy is extramized in accordance to the q-generalization of the Pesin's theorem (Tsallis, 2004b). The $\boldsymbol{q}_{sens}$ parameter characterizes the intermittent turbulence character of a spatially distributed complex and non-extensive system such as the space plasma systems.

2.2 Non-extensivity and intermittent turbulence

After all, the extramization of q-entropy creates the hierarchical and multiscale complexity of phase space. This is manifested as spatial intermittent NESS in the space plasma systems. The intermittent turbulence profile of the spatial distribution of space plasma streams is produced by the complex structuring of the system phase space. The phase space approach used in the nonlinear dynamics can reveal signatures of strong inhomogeneity, including a complex, multi scale and hierarchical system of islands and stochastic sea of islands that creates the stable nonrandom orbitals immersed in the stochastic sea of the chaotic phase space. Cantorus or cantori are boundaries between the islands or the system of islands and the stochastic sea, which can create trapping of the dynamics and creates movements of the dynamics perpendicular to the boundaries of the islands. Cantorus can be imagined as a fractal curve or line that includes an infinite number of gaps immersed in the stochastic sea of phase space, corresponding to the nonlinear dynamics. All points of the cantorus belong to the same orbit if the initial condition has been chosen from the cantorus manifold. Every island can be enclosed within an infinite set of cantori, the system of which can produce trapping or sticking ("stickiness") and flights of dynamics as the complementary features of anomalous diffusion of the nonlinear dynamics in the complex phase space. This is the meaning of complex or strange dynamics of complex and nonlinear systems, which produces the intermittency character of anomalous diffusion in the phase space or the physical space in the case of nonlinear and distributed dynamics as one of cases of space plasma dynamics. Moreover, strange dynamics creates multiscale and multifractal structures in the phase space or the physical space. Strange dynamics is also the fractional dynamical manifestation of the fractal topology structures, of percolation states and non-extensive statistics in the phase space. The fractional dynamics in the phase space is caused also by non-ergodic and intermittent self-similar hierarchies of islands, which can create spatial-temporal coherence and intermittency, through Levy type processes (Zaslavsky, 2002). Furthermore, the complex phase space is related to singular measures, singular (irregular) functions of space and time (fractal functions), as well as to scale invariance properties and multiscale interaction, causing long range correlations and hierarchical structures (Schlesinger et al. 1987, 1993; Schlesinger 1988; Arneodo et al. 1995; Zalsavsky, 2002).

The *q*-extended statistical mechanics of Tsallis and the Tsallis*q*-distributions correspond to general power law probability functions in phase-space with local singularities (a) related to the singularity spectrum functions *f*(a*)* and the generalized fractal dimension spectrum functions $D_{\bar{q}}$ (Theiler, 1990; Arneodo et al., 1995).According to this general theory, the non-Gaussian multi-scale correlations in the plasma system can create the intermittent multi-fractal structure of the phase space reflected also in the physical space multi-fractal distribution of the turbulent plasma dissipation field. The multi-scale interactions in the non-equilibrium critical space plasma NESS create the heavy tail and power law probability distribution functions, obeying the *q*-entropy principle. According to the non-extensive theory, the singularity spectrum of critical space plasma NESS is produced by the extremization of Tsallis *q*-entropy.

The Experimental Verification of Space Plasma Complexity

3.1 Magnetospheric Dynamics

In Pavlos et al. (Pavlos et al., 2011, 2012a), we used in situ spacecraft measurements of the bulk plasma flow in the Earth's magnetotail during two certain periods. The first period corresponds to a quiet plasma state, which is compared with the plasma state during the second period of superstorm geomagnetic activity. The q-statistics of Tsallis was estimated for the q-triples as measured from the magnitude of the bulk plasma flow in the magnetotail plasma sheet for the quiet and the active plasma states. The correlation dimension and the Lyapunov spectrum were also estimated for the two periods. The results showed the non-extensive intermittent SOC character of the magnetospheric dynamics during quiet periods. Also, as the magnetospheric plasma passes to strong superstorm active states it was shown the increasing of the non-extensive statistical character, including further development of long-range correlations and global magnetospheric self-organization. These results clearly indicate the existence of low-dimensional strong magnetospheric chaos related to strange attractor magnetospheric dynamics during the superstorm active periods. Comparison of the results obtained for the superstorm period and those found for the quiet period indicates a magnetospheric phase transition from a q-metaequilibrium state characterized by weak chaos, probably produced by high-dimensional SOC process, to a q-metaequilibrium state of strong low dimensional magnetospheric chaos. The quiet magnetospheric state at the edge of chaos corresponds to a second order thermodynamic phase transition process while the superstorm active state can be related to a first order thermodynamic phase transition. The non-extensive statistical character of the magnetospheric plasma that was shown by the analysis of in situ collected data indicates the long-range correlations existing in the magnetospheric plasma dynamics at the q-metaequilibriumthermodynamical states. The non-extensive character of the magnetospheric plasma was observed to be more intensive drying the second period of strong self-organization processes as it was shown by the estimation of Tsallis q-triplet.

3.2 Solar Wind dynamics

During coronal mass ejection (CME) events observed in the solar wind as interplanetary coronal mass ejections (ICMEs), we detected a strong non-extensive statistical character of the solar wind and magnetospheric plasma, as presented in the studies (Pavlos et al., 2015, 2016). During the period of the interplanetary shock passage through the Earth's orbit, the estimation of *q*-triplet of Tsallis showed clearly the strengthening of the non-extensive character, causing increase in q_{stat} and q_{sen} with simultaneous decrease in

q_{rel}. The increase of q_{stat} and q_{sen}iscaused by the enforcement of self-organization processes, creating the increment of the rate of *q*-entropy production.Also, the enforcementof self-organization processes, corresponds to decrement of q-entropy value and enhancement of multifractality-intermittency profile and development of non-local long-range correlations. The decrease of q_{rel} parameter is related also to the decrease of the relaxation time caused by the self-organizing and ordering process in agreement with the enhancement of the non-extensivity of the plasma system.

The study of the distant terrestrial magnetotail plasma showed properties similar to those observed in the distant interplanetary solar wind during the shock passage period. This is a significant result, indicating that during strong shock passage events and strong non-extensivity dynamics, both the magnetospheric and solar wind plasmasdemonstrate similar physical behaviors.

The space plasmas clearly reveal phase transition process, as the *q*-triplet of Tsallis changes drastically during shock events. Moreover, the observed changes of the *q*-triplet values indicate that the multifractal-intermittent character of the space plasmas phase transition, corresponds to a topological phase transition processes.

Finally, the non-extensive self-organization process of the space plasmas seems to be stronger far away from the Earth, outside the terrestrial magnetosphere or in the far terrestrial magnetotail. This result reveals that the strong interaction of space plasmas with the Earth bowshock reduces somehow the non-extensivity and the multifractality-intermittency of the system.

The phenomenology of the solar wind CME events, as it was summarized previously, reveals clearly the existence of a dynamical non-equilibrium and topological phase transition process related to the solar wind shock passage events and to the development of NES states (NESS) according to Zelenyi and Milovanov theoretical concepts (Zelenyi and Milovanov, 2004). This non-equilibrium phase transition process includes the transition of the original solar wind complex quiet state to states, charachterized by the enhancement of self-organization and intermittency. Although there are strong theoretical arguments, suggesting that the solar wind phase transition process and, more generally, the nonlinear turbulence solar wind processes, originate at the solar surface, the results of this study reveal that the solar wind plasma process exhibits certainly internally dynamical effects related to the observed transformation of the non-extensivity character and the enhancement of the self-organization process.

3.3 Sunspot dynamics

In Pavlos et al. study (Pavlos et al., 2012b), we used the SVD analysis in order to discriminate the dynamical components, underlying the sunspot index time-series. After that, we applied an extended algorithm for the nonlinear analysis of the original sunspot index time-series, its $\boldsymbol{V}_1$(first) SVD component and the signal $\boldsymbol{V}_{2\text{-}10}$ composed from the sum of the higher SVD components. The analysis was expanded to the estimation of: a) Flatness coefficients as a measure of Gaussian, non-Gaussian dynamics, b) The $q-$triplet of Tsallis non-extensive statistics, c) The correlation dimension, d) The Lyapunov exponent spectrum, e) The spectrum of the structure function scaling exponent, f) The power spectra of the signals. The results showed clearly:

1. The non-Gaussian and non-extensive statistical character of the solar dynamics underlying the sunspot index time series.
2. The intermittent and multifractal turbulent character of the solar plasma convection at the photosphere and the underlying convection zone.
3. The phase transition process between different dynamical profiles of the outer solar plasma (convective zone and photospheric region of the sun).
4. Clear discrimination between the high dimensional self-organized critical (SOC) solar state dynamical and the low dimensional dynamical chaotic solar state.
5. Novel agreement of the Tsallis theory predictions concerning the ***q***-entropy principle and the experimental estimation of solar intermittent turbulence indices corresponding: a) to the multifractal scaling exponents spectrum f(a), b) the generalized dimension spectrum Dq and c) the structure function scaling exponents spectrum J(p).

The noticeable difference between theoretically predicted spectra f(a), Dq, Jp can be explained by the fact that the High Dimensional (HD) part of the solar turbulence was used for the theoretical estimations. The theoretical predictions can be improved if calculations based on Tsallis theory are enriched by including the magnetic part of the plasma turbulence.

Moreover, significant and novel results included in this study concerns the following characteristics of the solar activity:

1. Intermittent turbulence of the solar convection zone
2. Strange nonlinear Solar dynamics and manifestation of q-extensive statistics according to Tsallis theory
3. Solar plasma self-organized critical (SOC) process
4. Solar plasma low dimensional chaotic dynamics
5. Solar plasma phase transition process in the convection zone

As we present in the next section, these results can be used for the theoretical understanding of solar plasma dynamics by using modern theoretical concepts as the fractal generalization of dynamics and nonequilibrium thermodynamic. This indicates the creation of a novel and universal modern theoretical framework of understanding space and cosmic plasma dynamics.

3.4 Solar Flares

In Karakatsanis et al. (Karakatsanis et al., 2013), we used the SVD analysis also in order to discriminatediscrete dynamical components underlying the solar flare index time-series. Similarly, to the sunspot dynamics, we applied an extended algorithm for the nonlinear analysis of the original solar flare index time-series, its V_1(first) SVD component and the signal V_{2-10} composed from the sum of the higher SVD components. We have found the following significant results concerning the solar atmosphere.

The non-Gaussian and non-extensive statistical character of the low solar corona dynamics underlying the solar flare index time series.

1. The intermittent and multifractal turbulent character of the solar low corona system.
2. The phase transition process between different solar flare dynamical profiles.
3. Novel agreement between different turbulence indices concerning entropy principle and the solar flare intermittent spectra: D(q), J(p) and f(a).
4. Clear discrimination of the solar flare dynamics from sunspot dynamics through the q-triplet of Tsallis and structure functions exponent spectrum.

The results of this study indicate clearly the non-Gaussianity and non-extensivity, as well as the multi-fractal and multi-scale dynamics of the solar flare process. According to these results, one can suggest the existence of a new mechanism of anomalous kinetic and magnetic energy dissipation and anomalous charged particle acceleration at the solar flare regions. This mechanism can be characterized as fractal dissipation - fractal acceleration mechanism, as the regions of dissipation – acceleration corresponds to fractal fields-particles distributions. It is also important to notice the similarity between the low solar corona dynamics and the dynamics of the solar convection zone reflected in the phase transition process from high dimensional SOC state to low dimensional chaos state. However, the discrimination of the two dynamical systems is possible by following the differentiation of the various dynamical characteristics.

3.5 Cosmic stars and cosmic rays

The nonextensivitycharacter of the dynamics of space plasmas,different than the magnetospheric or solar system was observed also by Pavlos et al. (Pavlos et al., 2014), in signals obtained from cosmic stars and cosmic rays time series.

Theoretical Estimations of Space Plasma Complexity. Theoretical Interpretations of Data Analysis and Results

The results of this study, can be better understood in the framework of modern theoretical concepts concerning non-extensive statistical mechanics (Tsallis, 2009), fractal topology (Zelenyi and Milovanov, 2004), turbulence theory (Frisch, 1995), strange dynamics (Zaslavsky, 2002), percolation theory (Milovanov, 1997), anomalous diffusion theory and anomalous transport theory (Shlesinger et al., 1993; Milovanov et al., 2001), fractional dynamics (Zaslavsky, 2002; Tarasov, 2005, 2006, 2013) and non-equilibrium RG theory (Chang et al., 1992).

4.1 Fractional Calculus

The fractal-multifractal structure of the space plasma systems indicates the necessity generalization of the classical MHD or Maxwell-Boltzmann plasma equations for the description of field-particle dynamics to the fractional dynamics of the space plasma systems.That is the space-time functions of the physical magnitudes are irregular functions, produced by the dynamics on the phase space structures. Moreover, the smooth differentiable nature of space-time functions and probability distributions at the macroscopic picture of physical plasma processes is a natural consequence of the Gaussian microscopic randomness. That is the microscopic Gaussian randomness is transformed into the macroscopic, smooth and differentiable processes, according to the classical CLT. Moreover, the classical CLT is related to the condition of microscopic and macroscopic time-scale separation, where at the long-time limit the memory of the microscopic non-differentiable character is lost. On the other hand, the *q*-extension of CLT induces the nonexistence of time-scale separation between microscopic and macroscopic scales as the result of multiscale global correlations, which produce fractional dynamics and singular functions of spatio-temporal dynamical physical variables.

The non-local character is evident in both cases of fractional derivative and integral on a fractal set. The non-local character of fractional calculus is related to multiscale and self-similar character of the fractal structure. The fractional extension of integral and differential calculus can be used for the description of the non-local multiscale phenomena described by fractional Maxwell's Equations (fME), or the fractional Magnetohydrodynamics (fMHD) of fractal plasma states, or the fractional Fokker-Planck Equation (fFPE) of fractal media (Tarasov, 2005, 2013). The solution of the fractional equations correspond to fractional non-differentiable singular self-similar functions, as we can observe in the experimental data. Generally, fractional differential integral equations have as solutions non-differentiable (singular) spatio-temporal distribution functions of physical magnitudes.

4.2 Anomalous diffusion and strange dynamics

Nonlinear dynamics can create fractal structuring of the phase space and global correlations in the nonlinear system. For non-extensive systems the entire phase space is dynamically not entirely occupied (the system is not ergodic), but only a scale-free –like part of it is visited yielding a long-standing (multi)-fractal-like occupation. According to Milovanov and Zelenyi (2000), Tsallis entropy can be rigorously obtained as the solution of a nonlinear functional equation referred to the spatial entropies of the subsystems involved including two principal parts. The first part is linear (additive) and leads to the extensive Boltzmann-Gibbs entropy. The second part is multiplicative corresponding to the non-extensive Tsallis entropy referred to the long range correlations. The fractal – multifractal structuring of the phase space makes the effective number W_{eff} of the possible states, probability of which is non-zero, to be smaller than the total number of states ($W_{eff} < W$). This is the statistical manifestation of self-organization process.

According to Zaslavsky (2002), the topological structure of phase space of nonlinear dynamics can be highly complicated due to trapping and transitions of the dynamics through a self-similar structure of islands. The island boundary is sticky making the dynamics to be locally trapped and "stuck". The set of islands is enclosed within the infinite fractal set of cantori causing the complementary features of trapping and flight being the essence of strange kinetics and anomalous diffusion.

The *q*-statistics of Tsallisis produced by the meta-equilibrium solutions of the FFPK equation (Tarasov, 2005; Tsallis, 2009). Also, the metaequilibrium states of FFPK equation correspond to the fixed points of Chang non-equilibrium RNG theory for space plasmas (Chang, 1992; Zaslavsky, 2002). The anomalous topology of phase space dynamics includes inherently the statistics as a consequence of its multiscale and multifractal character. From this point of view, the non-extensive character of thermodynamics represents a kind of unification between statistics and dynamics. From a wider point of view, the FFPK equation is a partial manifestation of a general fractal extension of dynamics. According to Tarasov (Tarasov, 2005), the Zaslavsky's equation can be derived from a fractional generalization of the Liouville and BBGKI equations. According also to Tarasov (Tarasov, 2005, 2006), the fractal extension of dynamics including the dynamics of particles or fields is based on the fact that the fractal structure of the spatially distributed matter (particles, fluids and fields) can be replaced by a fractional continuous model. In this generalization the fractional integrals can be considered as approximations of integrals on fractals. Also, the fractional derivatives are are created by development of long range correlations and localized fractal structures. In this meaning, the solar dynamo, which creates the solar and solar wind magnetic field, must be based at the extended fractal plasma theory, including anomalous magnetic transport and diffusion, magnetic percolation and magnetic Levy random walk. Also from the more extreme point of view, the fractal environment for the anomalous turbulence dissipation of magnetic field and plasma flows is the fractality of the space-time itself according to Shlesinger et al. (Shlesinger at al., 1993). That is far from equilibrium the microscopic fractality of space-time can create the macroscopic fractality of matter and field distributions. Finally, we can conclude that the solar wind phase transition process can be explained by the topological phase transition process of the attracting set in the phase space of the solar wind dynamics (Milovanov and Zelenyi, 1999; Zelenyi and Milovanov, 2004). According to the theoretical results of Chang's theory, the stochastic solar-solar wind and magnetospheric plasma system can exhibit low dimensional chaotic or high dimensional SOC like behavior, including fractal or multifractal structures with power law profiles. The power laws are connected to the near criticality phase transition process which creates spatial and temporal correlations as well as strong or weak reduction (self-organization) of the infinite dimensionality observed to a spatially distributed system. First and second phase transition processes can be related to discrete fixed points in the affine dynamical (Lagrangian) space of the stochastic dynamics. The SOC-like behavior of plasma dynamics corresponds to the second phase transition process as a high dimensional process at the edge of chaos. The process of strong and low dimensional chaos can be caused by a first order phase transition process. The probabilistic solution of the generalized Langevin equations may include Gaussian or non-Gaussian processes as well as normal or anomalous diffusion processes depending upon the critical state of the system.

From this point of view, a SOC or low dimensional space plasma intermittent chaos or distinct non-extensive *q*-statistical states with different values of the Tsallis *q*-triplet depends upon the type of the critical fixed (singular) point in the functional solution space of the system. When the stochastic system is externally driven or perturbed, it can transfer from a particular state of criticality to another characterized by a different fixed point and different dimensionality or scaling laws. Thus, the classic SOC theory could be considered as a special kind of critical dynamics of an externally driven stochastic system. Furthermore, SOC and low dimensional chaos can coexist in the same dynamical system as a process manifested by different kinds of fixed (critical) points in its solution space. Due to this fact, the space plasma dynamics may include coexistence of high dimensional SOC process or low dimensional chaos or other more general dynamical process corresponding to various *q*-statistical states.

CONCLUSIONS

In this paper, we have summarized experimental and theoretical results concerning Tsallis statistics and space plasmas complex dynamics. In particular, the estimation of various parameters, concerning Tsallis statistics and intermittent turbulence, revealed dynamical non-equilibrium phase transition processes and percolation topological phase transition indicated for the plasma dynamics at different regions of space (Zelenyi and Milovanov, 2004). In general, the non-equilibrium dynamics and phase transition processes include the transition of the original plasma complexity from calm state to shock or storm-substorm states, along with enhancement of self-organization, non-extensivity and intermittency.

The aforementioned results indicate the inefficiency of classical MHD or plasma statistical theories based on the classical central limit theorem used to explain the complexity of the solar plasmas dynamics, since these theories include smooth and differentiable spatial-temporal functions (MHD theory) or Gaussian statistics (Boltzmann-Maxwell statistical mechanics). The differentiable nature of smooth distribution of the macroscopic picture of physical processes is a natural consequence of the Gaussian microscopic randomness which, through the classical CLT, is transformed to the macroscopic, smooth and differentiable processes. The classical CLT is caused by the condition of time-scale separation, where at the long-time limit the memory of the microscopic non-differentiable character is lost. On the contrary, our results in a series of studies during the last two decades show the presence of non-Gaussian non-extensive statistics with heavy tails of probability distribution functions, which are related to the *q*-extension of central limit theorem. The *q*-extension of CLT induces the non-existence of time-scale separation between microscopic and macroscopic scales as the result of multi-scale global correlations.

These multi-scale global correlations are the basis for the fractal, multi-fractal structure of the solar and space plasma systems, producing fractional dynamics which can be described by the singular character of the spatio-temporal dynamical physical variables. Thus, a generalization from the classical field-particle dynamics of plasma theory towards the fractional dynamics of the space plasma system is needed. The fractional extension of integral and differential calculus can be used for the description of the non-local multi-scale phenomena described by Maxwell's equations, the magneto-hydrodynamics of fractal plasma states, or the plasma kinetic fractional equation of fractal media. Therefore, from the experimental data analysis results of this study and the theoretical framework of fractional dynamics, we can conclude that in each considered case the space plasma is a globally hierarchical, self-similar and scale invariant physical system that shows signatures of nonlinear and non-local internal fractional dynamics and fractional energization, maintaining the hierarchical structure of the intermittent turbulence. In parallel, the solar wind can reflect the fractional dynamics of the solar convection zone and solar photosphere. The nonlocal character of the fractional wave equation on fractals is generated by the correlated and coherent character of local dynamical events on a self-similar structure. The localization of waves on fractals (fractons) can be used for the explanation of local variation of solar wind characteristics, especially during shock events. In this direction, the solar wind an even everywhere in plasma systems we can observe fracton excitations and fracton dynamics where fracton formations are waves on fractal structures. Fracton dynamics can cause the oscillations of statistical parameters observed during space plasma phase transition.

In addition, the physical interpretation of our results indicate the possibility for the space plasma system phase to experience transitions from a weak non-equilibrium (quasi)-stationary state (NESS) to a strong NESS as the outcome of cluster interaction in the interplanetary space, as well as at the solar corona source of the solar wind. These states (NESS) can have the topology of a percolating fractal set, including multi-scale interactions of fields and particles (currents) and can be related to the simultaneous development of numerous instabilities interfering with each other. Thus, the solar wind plasma complex state is produced by the stabilization near the turbulent NESS identified with the generalized symmetries of a fractal disk diffeomorphism to a fractal set at the percolation threshold. The structural stability of the NESS as a symmetric turbulent phase is maintained due to multi-scale correlations creating the existence of local extremes of the free energy. According to this description, the solar- solar wind, or the magnetospheric plasma system can exist at distinct fixed points (different NESS) in the parameter space corresponding to the undisturbed and active states associated with passages of interplanetary shocks. Also the development of the solar wind shock or the magnetospheric substorm events corresponds to global dynamics phase transition process from the calm period fixed point of the Renormalization Theory to the shock or storm-substorm period fixed point in the space plasma dynamical parameter space.

Summarizing, Tsallis *q*-entropy principle can reliably explain the solar, solar wind and magnetospheric self-similar hierarchical turbulent structuring and phase transition processes presented in this study. Space plasmas, as any other complex system which are far from equilibrium, can reveal metaequilibrium stationary states (NESS) as critical percolation states. These non-equilibrium states, similar to Boltzmann-Gibbs thermodynamical meta-equilibrium states, can be produced as the system tends to obtain extremization of Tsallis *q*-entropy (S_q). The quantitative change of the non-extensive Tsallis statistics of the solar wind's system can be explained by the renormalization group theory (RGT) change of the fixed points (NESS) in the dynamical parameter space of the space plasma dynamics. The internal mechanism of this behavior is the anomalous diffusion process in the physical space or the anomalous random walk in a hierarchical and multifractal structured phase space. The dynamics in the multifractal phase space or physical space is described by the fractional equations (e.g Langevin and the FFPK equations). Moreover, we conjecture that the metaequilibrium stationary states can be obtained also as the fixed points of a fractional renormalization flow equation in a fractal parameter space. Also, the hierarchical, self-similar, multiscale and multifractal structure of the solar wind plasma system at critical percolation and intermittent turbulent states can be described by the solution of the fractional MHD Langevin equation, as the *N*-point correlation functions related to the functional derivative of the *q*-partition function Z_q defined in the framework of non-extensive Tsallis statistical mechanics-thermodynamics.

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