Solar Gravitational Multipolar Moments: an up-to-date pedagogical approach. Implications for stellar properties. An Analytical Overview.

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Abstract.
Solar gravitational multipolar moments have not been yet extensively analysed. However, they are at the crossroads of solar physics, solar astrometry, celestial mechanics, and General Relativity. Their values reflect the physics of solar models: non-rigid rotation, solar latitudinal rotation, solar-core properties, solar-cycle variations, and structure evolution. Their temporal variations are still often neglected; they are yet an essential aspect for constraining solar-cycle modelling or solar-evolution theories. They induced planet-planet inclinations in multi-transiting systems gravitating in the neighbouring of a star, leading to key issues future studies. This paper is devoted to an analytical analysis; a second part will address a helioseismology analysis.

Key words: sunquakes, solar flares

1. Introduction.

It has been suggested by Dicke (1974) that the measured excess of solar oblateness over the oblateness due to the surface rotation alone might be due to the existence of solar gravitational moments, themselves inter alia, due to a rapidly rotating solar core. This implies the knowledge of the mass distribution inside the Sun, and to first order, the gravitational quadrupole moment. The question is not completely solved and remains of great astrophysical interest, if, for example, the core is a fossil remnant of the rotation of the young Sun. The surface rotation of young solar-type stars, slightly more massive that the Sun decreases with age. If the surface rotation decreases, the central question is why a rapidly rotating core is still left in the solar/stellar interior? Another issue concerns the inclination of the planets inside our solar system, directly linked with the solar quadrupole moment. This point has been generalized for exoplanets, to calculate the mutual inclination induced between two (or more) planets as a function of the stellar oblateness (Spalding and Batygin, 2016).

Despite several attempts, we are still ignoring the exact determination of the successive solar gravitational moments, usually designed by Jn. By carefully analyzing in the existing literature the different techniques and models aiming at detecting Jn for n = 2, Pireaux and Rozelot (2003) assign J2 to be ≈ (2.0±0.4)x 〖10〗 ^(-7), bearing in mind that J2 is constraint by the lunar librations to be less than 3.10−6 (Rozelot and Bois, 1997). Higher moments (n>2) are not known with enough accuracy to be reasonably used today in modeling the solar interior properties. Furthermore, we do not know their temporal dependence, even if some attempts have already been made. They are notoriously difficult to measure directly: detecting a so faint value is at the cutting edge of the current available techniques, even through space dedicated missions. Helioseismology may provide an indirect alternative, that indeed has been used. However, the analysis to infer their estimates requires the development of efficient inversion methods and algorithms. By contrast, the Sun's interior structure and dynamics affect our solar planets trajectories, Mercury at first. Confrontation of collected data from spacecraft orbit determination with computed ephemeris by various institutes, are well-

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suited to determine J2 as a sub-product of the crossed analysis. It could be a mean to analyze their temporal variations, as today long-time series and space series of data are available.

To sum up, gravitational moments of a body in rotation are of fundamental interest, being at the crossroad of celestial mechanics, solar physics, solar astronomy-astrometry and General Relativity. Their accurate determinations, both theoretically and from observations lead to a better knowledge of the internal properties of the star, down to the core. By their external influences, they constrain planet-planet inclinations in multi-transiting systems gravitating in the neighboring of the rotating body -a forward thinking issue.

2. Effects of rotation upon the internal structure of a self-gravitating body.

Let us consider a single star that rotates along a fixed direction in space, with an angular velocity ω. The question we are faced is to determine its external shape. First assume that, for ω = 0, the star is a gaseous body in gravitational equilibrium. In tridimensional space, for such a fluid mass at rest, it has been demonstrated that the sphere is the unique solution to the problem of hydrostatic equilibrium. The issue complicates when the initial (perfect) spherical body is set rotating at an angular velocity ω. Axial rotation modifies the shape equilibrium configuration by adding a centrifugal acceleration term, breaking the spherical symmetry. The problem is then to determine the outer shape of the star, which is more complex than a simple ellipsoid, –then called spheroid – due to its stratification in density and the non-uniform distribution of the velocity rates of the matter at different latitudes and in depth. As a result, the exact equilibrium shape will show distortions at the free boundary, characterized by shape coefficients sn (related to asphericities coefficients as we will see in paragraph 2.3.1). Assuming spherical symmetry, the free surface is thus described as

$$r(\theta) = R_{sp}\left[1+\sum_{n=0} s_{2n} P_{2n}(\cos\theta)\right], \quad (1)$$

where r is the vector radius, Rsp the radius of a sphere of equivalent volume of the body, θ the polar angle (colatitude); P2n(cosθ) are even Legendre polynomials. Such a development of the radius of a rotating body is of interest as the photospheric radius is one of the fundamental parameters governing the radiative equilibrium of a star. Moreover, it is straightforward to see (by writing r(θ=0°) - r(θ=90°)) that the flateness f (in the general case of a fluid in rotation) is a linear function of the asphericities terms:

$$f = -(3/2)s_2 - (5/8)s_4 - (21/16)s_6 - (\ldots)s_n.$$

2.1. Enhancing the historical diagnostic.

Considering the Earth as a rotating ellipsoid in uniform rotation ω, Newton gave in his "Philosophiae Naturalis Principia Mathematica" in 1687, an approximate formulation of the Earth's flattening f, as a function of surface gravity g_s: $f=5/4\ \omega^2 . R_{eq}/g_s$, where R_{eq} was the equatorial radius (of the body). Huyghens, in 1690, reformulated the flattening in the form $f=1/2\ \omega^2 . R_{eq}/g_s$, still commonly used as a first approximation.

Since then, question of the flattening of rotating bodies has been tackled by several authors and by several means. The "classical theory" goes back with Maclaurin (1742) and Clairaut (1743) who treated the case of a uniform ellipsoid. Then Jacobi (1834), Riemann (1860) and Chandrasekhar (1965, 1966) processed the spheroids case, Poincar´e (1891) those of a uniform torus, and Jeans (1914) polytropic cylinders. Solutions obtained by these methods are consistent only if the distributions of density and angular velocity are restricted to certain special well determined laws. However, the method seems not limited to slow rotation, and can goes to more rapidly rotation, but only when the quantity $|T/W|$, ratio of the kinetic energy of rotation (T) to the (negative) gravitational energy ($-W=|W|$) may approaches 0.5 (Lyttleton, 1953) for the most flattened Maclaurin spheroids. A star is thus considered to be "slowly rotating" if $|T| \ll |W|$, without regard to the equatorial velocity or the ratio of centrifugal force to gravity at the surface. "Slowly rotating" stars may have surface layers which are severely distorted by rotation, with consequences of the distribution of the emitted light (Ostriker and Mark, 1968). Since then, an extensive literature on the equilibrium structure of rotating bodies has been produced which can be roughly divided into three main approaches.

•The first and older one introduces level surfaces to describe the limb shape of the rotating body. The problem is carried to successively higher approximations by transforming the expressions for the gravitational potential to level surfaces. An extension of this method has been made by Bruns (1878), Radau (1885), Darwin (1899), Wavre (1932), Milne (1923), Chandrasekhar (1933) or Chandrasekhar and Lebovitz (1962) who treated objects with equipotentials which deviate only slightly from oblate surfaces. Extensive solutions have been widely studied by Modolenvsky (1988), who reached to the conclusion that the physics of equilibrium figures (density ρ, gravity g, pressure p) are completely determined by the geometrical stratification of the rotating body (Modolenvsky calls "stratification" the geometry of the successive equilibrium surfaces of a rotating

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body when "looking" progressively from the surface to the interior). Important contributions were latter on conducted by Hubbard (1975), Zharkov (1978), Moritz (1980), and many others. This theory, completed with results obtained from the advent of artificial satellites, has been widely used in geophysics and is still used in specific cases, such as for planets, with an incredible accuracy (ex: J2 Mars = 1.860718 10−3 according to Yuan et al. (2002), from a 75th degree and order model).

•A second method avoids the difficulties encountered in accurately determining the potential, by treating it as a given quantity. For instance, in the Roche model (1849 for his first paper), the potential is taken to be that of a point mass, an approach taken again more recently by Ess´en (2014) who deduced accurate analytical results for the Sun and planets connecting the oblateness, the gravitational quadrupole moment and the angular velocity parameters.

•The most straightforward approach has been first taken by James (1964) and Stoeckly (1965), who numerically integrate the partial differential equa- tions of equilibrium. The method has no obvious limitations and can be made as precise as is desired, although an accurate representation of the equidensity surfaces requires a fairly dense grid of points and correspondingly lengthy com- putations. The difficult part of this problem –satisfying Poisson's equation in the interior with boundary conditions on a surface determined by the solution of the equations of equilibrium– can be solved by successive approximations. Closer to us, Maeder (1999) discussing the von Zeipel theorem generalized to account for differential rotation in the case of a "shellular" rotation law (i.e. when ω is constant on isobaric surfaces) has found that this differential rotation increases the oblateness. Mathis et al. (2018) reviewed the stars equilibrium forms, of arbitrary structure, distorted by rotation and tides. Up-to-date rotating models have been produced, such as those produced by the ESTER code (Rieutord et al., 2016), in which structure and rotation are inherently coupled in two dimensions.

2.2. The Maclaurin's approach

The stability of the spheroidal figures of equilibrium of a rotating fluid was studied by many authors at the end of the 19th century (such as Dedekink in 1860, Jacobi in 1834 or Rieman in 1876). Maclaurin first established in 1742 that ellipsoids are stable or unstable according to whether their eccentricities are less than or greater than 0.9528. However, Rumiansev (1959) using a rigorous definition of the stability of figures of equilibrium proved that ellipsoids of revolution remain stable as long as their eccentricities are less than 0.8126.

Defining q by

$$q=\frac{3\omega^2}{4\pi\rho_m G}=\frac{\omega^2 R_{eq}^2 R_{pol}}{GM},$$

where G is the gravitation constant, R_{eq} and R_{pol} are respectively the equatorial and polar radius, ρ_m the mean density of the body and M its mass (to first order, $M=(4/3)\pi\rho R_{eq}^2 R_{pol}$), equations are closed by $q=\frac{3}{2l^3}\left[(3+l^2)\arctan l-3l\right]$ where l is the inverse of the second eccentricity of the ellipsoid $l=\sqrt{R_{eq}^2-R_{pol}^2}/R_{pol}$ related to its first eccentricity by $\epsilon=l^2/(1+l^2)$.

The above equation in q has one root $l_0=0.717$ corresponding to an eccentricity of the body $\epsilon_0=0.8126$. Moreover, for $l/2(\omega^2/\pi\rho G)=0.1871$ the axes a and b become equal and the Jacobi ellipsoid turns into an ellipsoid of revolution, which at the same time is also a Maclaurin ellipsoid. For $l/2(\omega^2/\pi\rho G)>0.1871$ tri-axial ellipsoids of equilibrium of a rotating fluid do not exist.

The Maclaurin spheroid is considered to be the simplest model of rotating ellipsoidal figures in equilibrium since it assumes uniform density. However, heterogeneous mass distribution has been studied using concentric Maclaurin spheroids (CMS). The method has been mainly developed by Hubbard (2012, 2013) and is consisting of a numerical hydrostatic scheme which decomposes a rotating celestial body into N spheroids of constant density (the initial Maclaurin spheroid would correspond to the case N = 1). For a given density profile, the gravitational potential of each spheroid (which is constant on the spheroid) is calculated. Then is computed on a self-consistently way the radius of each spheroid as a function of latitude. Such a method has been developed for solar system planets, Mars, Jupiter, Neptune or Uranus. The method is somewhat limited; to fulfill the Jupiter's observations of the gravitational moments for instance, at least 1500 spheroids must be used (Debras and Chabrier, 2017). It gives nevertheless very satisfactory results, the fit being up to the 3rd digit at least. The CMS method has never been applied to the Sun.

For a constant density oblate region, the Jn can be analytically determined (through what is called the "exact" Maclaurin solution), that are (Klioner, 2003; Hubbard, 2012; Panhans & Soffel 2014) :

$$J_{2n}=\frac{3(-1)^{n+1}}{(2n+1)(2n+3)}\left(\frac{l^2}{1+l^2}\right)^{n} \quad \text{or} \quad J_{2n}=\frac{3(-1)^{n+1}}{(2n+1)(2n+3)}(\epsilon)^{n} \qquad (2)$$

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where l is also given by $l^2=(R_{eq}^2/R_{pol}^2)-1]$. Note that l is related to f by $l^2=f^2-2f$ or conversely $f=1-\sqrt{1+l^2}$. Here f is the "true" oblateness $(R_{eq}-R_{pol})/R_{eq}$ (different from $f^* = (R_{eq}-R_{pol})$ sometimes used as "oblateness" .

As an example, taking $R_{eq}=695509.9835$ km and $R_{pol}=695504.0331$ km in such a way that $R_{\odot}=3Req2\times R_{pol}=695508.0000$km (astrometric accuracy!), it comes

$f = 8.56\times10^{-6}$, $q = 2.05\times10^{-5}$ (= 2.85 rad/s at the equator) and

$J2 = -3.42\times10^{-6}$, $J4 = 2.51\times10^{-11}$, $J6 = -2.39\times10^{-16}$, $J8 = 2.60\times10^{-21}$...

Such values are crude order of magnitudes, as the density is taken as constant (solar mean value) and the differential rotation is not taken into account. However, they show that solar gravitational moments are rather faint and are decreasing as far as the order n is increasing. Note that Ess´en (2014), using a point core model found the following estimates:

$q = 1.15\times10^{-5}$, $f = 6.3\times10^{-6}$ and $J_2= 3.7\times10^{-7}$.

One will notice that in all cases, J_n is of the order of f^n.

2.3. Analysis through the gravity field

As the outer equilibrium surface of a rotating star differs from sphericity, and even if in general the deviations are small, it results that the matter is not evenly distributed. The flattening of a star (or the Sun), for which we have a priori no knowledge of its stratification, limb boundary, planes of symmetry, transfer of angular momentum in a differentially rotating body, etc, implies to first order a bulge at the equator, and to higher orders deviations to sphericity. These excesses (or deficit) of mass are due to the existence of the gravitational moments J_n in the expression of the total potential of the body (sum of the gravitational potential and the potential of centrifugal forces). The even order n = 2 is called the quadrupole moment J2. Higher orders are called dodecapole J4 (sometimes also called octupole), hexadodecapole J6, etc...

2.3.1. The gravitational potential

The gravitational potential can be expressed as:

$$\Phi_g(P)=\Phi_g(x,y,z)=G\iiint \frac{dm}{l} = G\iiint \frac{\rho}{l}\,dv$$

where P (x, y, z) denotes the point at which Φg is calculated, dm is the mass element for which $\rho = dm/dv$ and $l=(x^2+y^2+z^2)^{1/2}$. The Laplace's Equation $\nabla\Phi_g = 0$, written in spherical coordinates (r, θ, λ) can be solved by a product of three functions, each of which depending on only one coordinate:

$\Phi g = -f(r)g(\theta)h(\lambda)$, whose solutions are:

$$\begin{cases} f(r) = r^n \text{ or } r^{-(n+1)} \\ g(\theta)=P_{nm}(\cos\theta) \\ h(\lambda)=\cos(m\lambda) \text{ or } h(\lambda)=\sin(m\lambda) \end{cases}$$

where n = 0, 1, 2, 3, ... , is called the degree and m = 0, 1, ... n, is called the order of the function under consideration. Pnm is the Legendre polynomial of degree n, for a given m.

In 1875, A.M. Legendre published his book "Sur l'attraction des sph´ero¨ıdes" in which he developed the gravitational potential in terms of a power series

$$V(r)=-GM/r+O(1/r^n) \quad (3)$$

where, again, M is the total mass and G the Gravitation constant

($G = 6.67259(85) \times 10^{-11}\ m^3kg^{-1}s^{-2}$). $O(1/r^n)$ indicates that for $r\to\infty$, this term tends to zero as $1/r^n$. Following this formalism, the gravitational potential $\Phi g(r, \theta, \lambda)$ can be developed under the series

$$\Phi_g(r,\theta,\lambda)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{1}{r^{n+1}}\left(C_{nm}P_{nm}(\cos\theta)\cos(m\lambda)+S_{nm}P_{nm}(\cos\theta)\sin(m\lambda)\right)\right],$$

which is also

$$\Phi_g(r,\theta,\lambda)=-\sum_{n=0}^{\infty}\left[\frac{1}{r^{n+1}}\sum_{m=0}^{\infty}\left[C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)\right]P_{nm}(\cos\theta)\right]$$

Since the first term n = 0 is nothing else but GM/r (see Eq. 3), it turns out that

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$$\Phi_g(r,\theta,\lambda)=-\frac{GM}{r}\left[1-\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(\frac{a}{r}\right)^n [C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)]P_{nm}(\cos\theta)\right]. \quad (4)$$

The coefficients Cnm are called the tesseral coefficients; for m = 0, Cn0 are the zonal coefficients; C20 (namely zonal harmonic of degree -2 order -0) is the dynamical flattening of the body (but not the flatness, as sometimes written, mainly in planetary ephemerides papers), and is usually designed by J2,Sun or if no ambiguity, more simply by J2. Note that the degree n is linked with the wavelength ς of the mass anomalies inside the body: n = 2πa/ς. The larger the n are, the finer is the resolution.

For the general case, the rotating body takes the shape of a spheroid, which can be described by successive parameters, f to first order (as above-mentioned), h and k for higher orders, and is either oblate (elongation along the equatorial axis, n = 1, m = 1 –called oblateness–), or prolate (elongation along the polar axis, n = 1, m = 2 –called prolateness–. Figure 1 visualizes the different cases. For the Sun (or stars), for which an axially symmetric distribution of the rotating matter can be assumed (not for the Earth), Snm = 0, m = 0, so that

Eq. 4 reduces to:

$$\Phi_g(r,\theta)=-\frac{GM}{R}\left[1-\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n}[C_{2n}]P_{2n}(\cos\theta)\right], \quad (5)$$

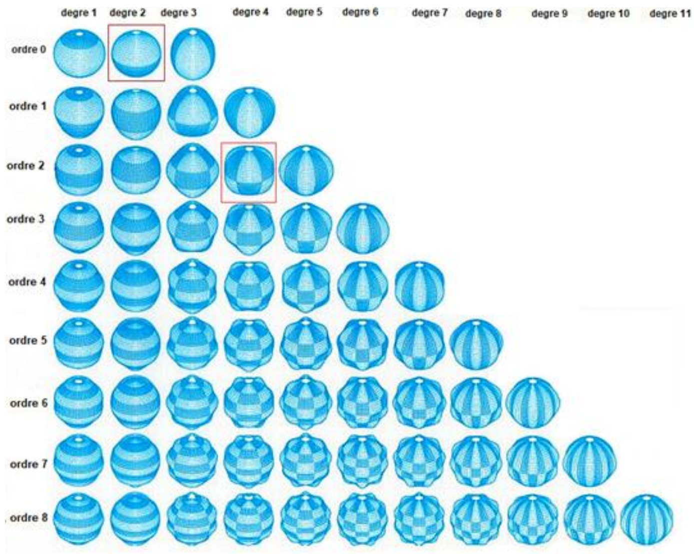

Figure 1: Laplace spherical harmonics showing the surface distortions of a ro- tating fluid body in the general case. The gravitational moment J2 is given by its degree n = 2 order m = 0, which is the dynamical flateness, c20. Note on the same way J4. (After R. Biancale, 2006 personal communication).

where C2n are the degree -n order -0 gravity field of the body under consider- ation. In the solar case, n = 2 is commonly called Sun's gravitational degree 2 zonal harmonic 0, or more simply J2 (which is not the oblateness but only an 'indicator').

The centrifugal potential

Φ_c is given by:

$$\Phi_c=\int_0^{r\sin(\theta)}\omega^2 r\sin(\theta)\,d(r\sin\theta) \quad (6)$$

for a body rotating at a velocity rate ω, which is, integrated over the sphere r = Rsp

$$\Phi_c=\frac{1}{2}\omega^2 R_{sp}^2\sin^2\theta=\frac{1}{3}\omega^2 R_{sp}^2[1-P_2(\cos\theta)] \quad (7)$$

•The case study for a uniform rotation ω0.

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From Eq. 1 and truncated to order 2, we get successively $1/r=(1/R_{sp})(1+2/3\, fP_2+O(f^2)$ and the radius of the sphere R_{sp}, which is determined for $P_2=0$: $R_{sp}=R_{eq}(1-1/3\, f)$. It results that the gravitational potential Φ_c can be expressed as function of R_{sp} as

$$\Phi_g=-GM/R_{sp}\,[1+2/3\, f+J_2 P_2]\text{, accurate up to } O(f^2).$$

The total potential is thus

$$\Phi=-GM/R_{sp}\,[(1+2/3\, q)+(2/3\, f-J_2-1/3\, q)P_2]\,, \qquad (8)$$

setting $q=(\omega^2 R_{sp}^3)/GM$ as previously seen.

If the solar ellipsoid is a level surface, Φ must be constant on it; thus, the coefficient of P_2 in Eq. 8 must vanish:

$$2/3\, f-J_2-1/3\, q=0$$

or

$$J_2=2/3\, f-1/3\, q \qquad (9)$$

Using previous numerical values, inferring no differential rotation, and taking $\omega=2.85$ µrad/s (i.e. the velocity rate at the equator) , we get $q = 2.059 \times 10^{-5}$, so that J_2 in this case and limited to order 2 is:

$$J_2=-1.16 \times 10^{-6}$$

•Higher orders.

Designing by $k(R_{eq})$ and $h(R_{eq})$ the second-order and third-order corrections to the equation of a spheroid whose semi-axes are R_{eq} and R_{pol}

$R_{pol}=R_{eq}(1-f)$ is

$$r(\theta)=R_{eq}\,[1-f\cos^2\theta-(3/8\, f^2+k)\sin^2 2\theta+1/4\,(1/2\, f^3+h)(1-5\sin^2\theta)\sin^2 2\theta+\cdots]. \qquad (10)$$

Hence it comes:

$$R_{eq}=s(1+1/3\, f+2/9\, f^2+8/15\, k+14/81\, f^3+26/105\, h+16/63\, fk) \qquad (11),$$

or conversely

$$s=R_{eq}(1-1/3\, f-1/9\, f^2-8/15\, k-5/81\, f^3-26/105\, h+32/315\, fk) \qquad (12).$$

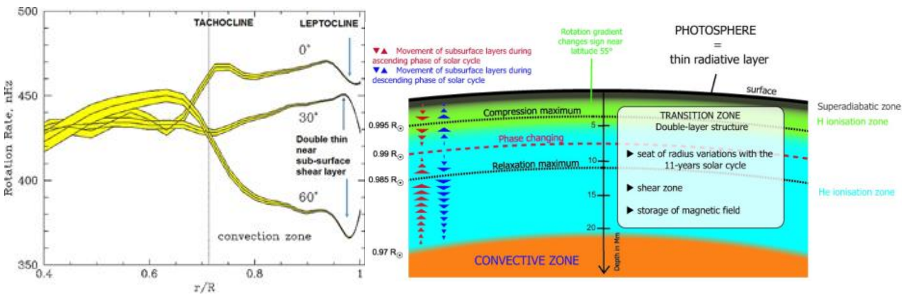

Figure 2: The differential rotation discovered at the surface of the Sun through spots is extended into the interior and is detected by helioseismology (left). Two main layers can be seen, the tachocline, seat of the differential rotation, and the leptocline, a near sub surface layer (NSSL) where significant physical phenomena seem to be anchored (right): non-homologous radius changes in time and in depth, shearing, disturbance of the turbulent pressure, constraints upon the magnetic field, processes of ionization, etc.

After some algebra, the analytical form of the Jn can be found as:

$$J_2=+[2/3\, f-1/3\, q-1/3\, f^2+8/21\, k(1+q)+3/7\, fq+40/147\, fk-50/294\, f^2 q-2/21\, h],$$

$$J_4=-[-4/5\, f^2-32/35\, k+4/5\, f^3-50/49\, f^2 q+3616/2695\, fk-4/7\, fq+208/385\, qk-192/385\, h],$$

$$J_6=+[8/7\, f^3-20/21\, f^2 q-160/231\, qk+128/77\, fk-80/231\, h]. \qquad (13)$$

(…)

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Note that J_4 is of the order of + 1.45 x 10-10 and J_6 of the order of -7.20 x 10-16.

3. Case of the differential rotation

The main difficulty comes from the differential rotation of the Sun, that is to say the variation of the rotation rate with latitude: the equatorial zones of the Sun rotate a little faster than the polar ones. This phenomenon has been detected immediately after the discovery of the sunspots when astronomers saw that their apparent motion took a little longer time to cross the solar disc at higher solar latitude than near the equator. The first major work on this subject was written in 1630 by Scheiner (Rosa Ursina sive sol) and since then, this physical phenomenon has never ceased to attract researchers mainly fortwo reasons: (i) its origin remains unsolved and (ii) it never stopped: in other words, there is no regulating process which would tend to standardize rotation over long periods of time.

The differential rotation is continuous in the sense that the rotation rate is slowly varying from the equator to the pole. This phenomenon goes further in- side, down to the tachocline, a transition region interfacing the radiative interior and the differentially rotating outer convective zone. Helioseismology measure- ments indicate that the tachocline is located at a radius of at most 0.70 times the solar radius, with a thickness of 0.04 times the solar radius. The rotation rate through the interior is roughly equal to the rotation rate at mid-latitudes (i.e. approximately in-between the rate at the slow poles and the fast equator). Below the tachocline, the rotation is more uniform, but increasing with depth (see paragraph 4.1).

Solar rotation has been investigated by a variety of techniques. Observed data can be fitted through a law usually taken as

ω = ω0 + ω1sin2(θ) + ω2sin4(θ) (15)

where ω0 is the angular velocity rate at the equator (and θ the heliographic latitude). The coefficient ω0 represents the equatorial rotation rate and the coefficients ω1 and ω2 measure the latitudinal gradient in the rotation rate, with ω1 representing mainly low latitudes whereas ω2 represents largely higher latitudes (Javaraiah 2009).

Equation 15 can be developed over the Legendre’s polynomials basis, which leads to

ω = (ω_0+1/(3ω_1)+1/(5ω_2))+ (2/(3ω_1)+4/(7ω_2)) P_2+8/(35ω_2 P_4) (16)

Putting 16 into 7, the centrifugal potential is of the form

Φ_C = -1/3 〖ω_0〗^2 〖R_eq〗^2 [A_0+A_2 P_2+A_4 P_4] (17)

The total potential Φ, is the sum of the gravitational potential and the centrifugal one, Φ_g and Φ_c, both depending on the radius r and the colatitude θ:

Φ(r, θ) = Φ_g (r, θ) + Φ_c (r,θ),

which can be expressed as

Φ(r,θ) = K[B_0+ B_2 P_2+ B_4 P_4 + ...] + O(f^n)]

On a level surface, Φ_g must be constant on it; thus, the coefficient of P_n must vanish, and the J_n can be derived from B_n= 0.

Numerical expression

Without removing generality, it is possible to compute J_n by remembering that in Eq. 10, it can be seen that the excess of its radius vector over that of the true ellipsoid is obtained at the colatitude for which the change occurs in the external contour shape, i.e. when the radial rotation gradient ∂ω/∂r changes in sign. Indeed ∂ω/∂r is < 0 at the equator up to θ≈(40-55)^° and is positive thereafter .

Adopting for instance the Javaraiah’s (2002) law, ω = 2.936 - 0.56 sin2(θ), it comes at θ = 46◦,

$J2 = -2.10\times10^{-7}$, $J4 = +1.45\times10^{-10}$ and $J6 = -5.22\times10^{-16}$

which can be compared to the values obtained by Armstrong and Kuhn (1999):

$J2 = -2.22\times10^{-7}$, $J4 = 3.84\times10^{-9}$.

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We can deduce the shape coefficients (limited to order 2) from $s2 = −J2 − (1/3) × q$ and $s4 = −J4 + (6/35) ×q$, which are

$s2 = -5.71×10^{−6}$, $s4 = 3.04 ×10^{−6}$

and again, can be compared to the values deduced by Armstrong and Kuhn (1999):

$s2 = -5.84 ×10^{−6}$, $s4 = 0.59 ×10^{−6}$

It can be checked that f, which is ≈ - (3/2) s2 - (5/8) s4, is thus $6.66 ×10^{−6}$ in the first case and $8.39×10^{−6}$ in the second one (to be compared with the adopted value of $8.56×10^{−6}$).

Lydon and Sofia (1996) examining 20 years of surface velocity field data obtained at Mount Wilson Observatory found the following estimates:

$J2 = -1.84×10^{−7}$, $J4 = +9.83×10^{−7}$ and $J6 = -4.0×10^{−8}$.

A full comparison of several data set has been made in Rozelot et al. (2009).

For the time being, we have no explanation for the discrepancy on the n = 4 term. Armstrong and Kuhn (1999) have provided a more sophisticated analysis, considering the non-uniform rotation over latitudinal cylinders. However, these authors have claimed that the quadrupole (and hexadecapole) limb shape are mildly inconsistent with current solar rotation models' and that the discrepancy will probably be removed by a better understanding of the solar core rotation. Progress on this problem will depend on improved measurements of the limb shape (through dedicated missions), but also by a better under- standing of the two-dimensional solar interior rotation rate. However, the J2 estimates, as well as their order of magnitudes, are in good agreement with these deduced from other methods. Thus Patern´o (1996), Godier and Rozelot (2001), Pireaux and Rozelot (2003), Eren and Rozelot (2020, weighted estimate) found respectively (in $10^{−7}$), -2.08 ± 0.14, -2.0 ± 1.4, -2.00 ± 0.40 and -2.17 ± 0.06. Pijpers (1998) found +2.18 ± 0.06; if the order of magnitude is respected, the sign is not, and this is not only a matter of convention: its value is questionable. From observations, the discrepancies found may be due to the temporal dependence of the gravitational moments as will see latter on. Thus, we emphasize the need to accurately measure the limb shape and its variation with time.

Second order analysis of the differential rotation.

The usual centrifugal potential Φ_c must be rewritten to consider the differential rotation on a physical point of view; in this case the problem is no longer conservative, and the star is baroclinic. However, we can postulate that the centrifugal force must derive from a potential, in such a way that it must be possible to find a function U which satisfies (Lefebvre 2003):

$$\vec{F}_{centrifugal} = -\vec{\nabla} U$$

At a depth rp, one can use an equation of the form

$$\Omega = \Omega_{pol} \left[1+\sum_{i=1}^{\infty} a_{2i} r_p^{2i} \cos^{2i}(\theta)\right]^{1/2} \qquad (18)$$

which derives from a potential. Using the solar Greenwich database that records the sunspots position as a function of time, Javaraiah and Rozelot (2002) have computed ai for i = 1 and i = 2, which are:

a2 = +0.442, a4 = +0.056 at the surface (rp = 1)

and ωpol = 2.399 µrad/s.

The reader will be able to verify the perfect adjustment of the two curves given by Eq. 15 and 18, using the numerical values given here. Fig. 3 shows the rotation rate with depth (from rp = 1 down to 0.75 $R_\odot$). The inversion of the radial gradient rotation rate can be seen at θ = 37◦ of latitude, within the leptocline. This mechanism signs the main difference with a stellar structural approach and put in evidence the key role of the solar gravitational moments J2 and J4.

Best known estimates of ω0, ω1 and ω2, according to several authors, are given in Table 2 published in (Rozelot 2003b). The following two examples differ from the way sunspots and faculae rotation were analyzed (coefficients in µrad/s):

ω0 = 2.913, ω1 = - 0.283 and ω2 = -0.269 (Kuiper 1972),

ω0 = 2.82, ω1 = - 0.33 and ω2 = -0.53 (Howard 1980).

Numerical values are of importance, as a small change could have a relatively large effect on the implied multipolar moments .

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The rotational potential Φ_c can then be expressed in terms of Legendre's polynomials

$$\Phi_c = -\chi \frac{GM}{R_{sp}} [A_0 + A_2 P_2 + A_4 P_4]$$

with χ defined as $\chi = \omega_{eq}^2 R_{eq}^3/GM$. (Note that q and χ are related in terms of each other and are expressed in terms of s0, s2, s4..., as:

$$q/\chi = (R_{eq}/s)^3 = 1 - 3/2\, s_2 + (3/2\, s_2^2 + 3 s_0 + 9/8\, s_2^2) + \cdots).$$ It turns out that

$$A_2 = [(-46832/135135\, a_4 - 3064/10395\, a_2 - 82/315) f^2 + (-160/693\, a_4 - 16/63\, a_2 - 20/63) f + (-8/63\, a_4 - 4/21\, a_2 - 1/3)] \quad (20)$$

and

$$A_4 = [(9552/25025\, a_4 + 77456/225225\, a_2 + 76/231) f^2 + (96/455\, a_4 + 256/1155\, a_2 + 8/35) f + (24/385\, a_4 + 2/35\, a_2)] \quad (21)$$

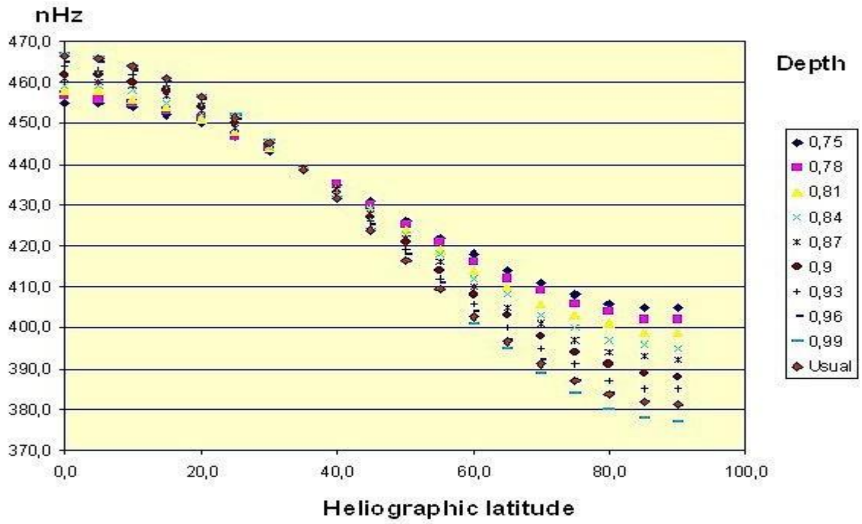

Figure 3: Differential rotation (velocity rate in nHz versus the heliographic latitude), for the solar case and according to a law deriving from a potential. The different depths are listed in the right box. One can perfectly see the inversion of the radial gradient of rotation at θ = 37◦ of latitude (Influence of the leptocline at 0.99 $R_\odot$. See Lefebvre et al., 2007

Writing that on the surfaces of constant potential Φ must not depend on the heliographic latitude, it comes after computations, and accurate up to $O(f^3)$

$$J_2 = 2/3\, f - 1/3\, f^2 + \chi A_2 - 26/21\, \chi A_2 f - 20/21\, \chi A_4 f + 521/2205\, \chi A_2 f^2 + 3350/4851\, \chi A_4 f^2 + (\ldots) \quad (22)$$

and

$$J_4 = 4/5\, f^2 + \chi A_4 - 36/35\, \chi A_2 f - 502/231\, \chi A_4 f + 8309318/3468465\, \chi A_4 f^2 + 4866/2695\, \chi A_2 f^2 + (\ldots) \quad (23)$$

where A2 and A4 are determined by the ai.

With the numerical values already given, one obtains A2 = -0.42(4638) and

A4= + 0.028(751) at $r_p = 1 R_\odot$. It is obtained in such a way J4 = (4.36±0.2) 10^{-7}.

Another approach is to develop the solar rotation described by Eq.15, by means of a set of disc-orthogonal functions

$T_1^0(\sin\theta) = 1$,

$T_2^1(\sin\theta) = 5\sin^2\theta - 1$, and

$T_4^1(\sin\theta) = 21\sin^4\theta - 14\sin^2\theta + 1$,

which leads to the following expansion:

$$\omega(\theta) = \bar{A} + \bar{B}(5\sin^2\theta - 1) + \bar{C}(21\sin^4\theta - 14\sin^2\theta + 1) \quad (24)$$

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The coefficients $\bar{A}$,$\bar{B}$ and $\bar{C}$ are free of crosstalk,(A)¯represents the ‘rigid body’ (or ‘mean’) component in the rotation, $\bar{B}$ and $\bar{C}$ are the components of the differential rotation. If the polynomial expansion is terminated at $\bar{C}$, the coefficients $\bar{A}$, $\bar{B}$, and $\bar{C}$, are related to the standard A, B, and C coefficients as follows:

$\bar{A}$ = A + (1/5)B + (3/35)C,

$\bar{B}$ = (1/5)B + 2/15)C, $\bar{C}$ = (1/21)C.

Using Legendre polynomials P0, P2 and P4 as a set of orthogonal functions, the differential rotation can be described by:

$\omega(\theta) = DP0 + EP2(\cos\theta) + FP4(\cos\theta)$. (25)

If the expansion is truncated at the third term, the coefficients D, E, and F

are related to the coefficients A, B, C in Eq. 25 as follows:

D = A + (1/3) B + (1/5) C,

E = (2/3) B + (4/7) C,

F = (8/35) C.

4. Discussion.

The accurate determination of the J4 term is not easy, as it can be seen from several papers already published on this question. Taking into account the role of the solar magnetic field, it has been found J4 = 6.291 10−7 (Ajabzirizadeh et al., 2008; error bars can be set up at 2.0 10−7).

These values are an order of magnitude less that those obtained through the theory of stellar structure (see Pireaux and Rozelot, 2005). In the helioseismology analysis, the kernels for the high-order multipole moments are more concentrated near the surface and may be responsible for the difference. However, our estimate is consistent with observations of Lydon & Sofia (1996), leading to J4 = 9.83 10−7 (and J2 = 1.84 10−7).

We think that the contribution of J4 – if its value is confirmed to be of the same order of magnitude than J2– is a key to explain solar observations. Let us report here the results obtained from space missions:

Emilio et al. (2007) reported a solar shape distortion using the Michelson Doppler Imager (MDI) aboard the Solar and Heliospheric Observatory (SOHO) satellite, after correcting measurements for bright contamination. It was found that the shape distortion is nearly a pure oblateness term in 2001, while 1997 has a significant hexadecapolar (J4) shape contribution (Fig. 4).

Antia et al. (2008) analyze on a timescale of the solar cycle, the variation of the angular momentum of the Sun and the associated variations in the gravitational multipole moments, by inverting helioseismic rotational splitting data (obtained by the Global Oscillation Network Group and by the Michelson Doppler Imager (MDI) on the Solar and Heliospheric Observatory(SOHO). They found a temporal variation in angular momentum at high latitudes (> 45◦) through the convection zone positively correlated with the level of solar activity, whereas at low latitudes it is anticorrelated, except in the top 10 % by radius where both are correlated positively.

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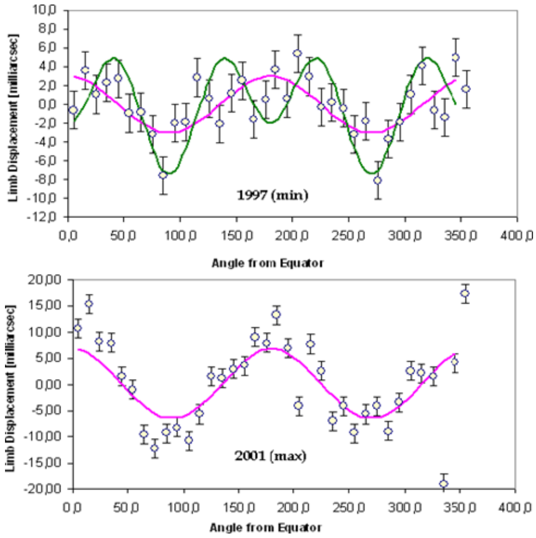

Figure 4: The solar shape variations in time deduced from SOHO. The oblateness contribution (linked to the quadrupolar term J2 term) is the only one in 2001, albeit a significant hexadecapolar (J4) contribution is predominant in 2007 (After Emilio et al., 2007).

The variation (in relative value, not in absolute one) is less than the one we found from the sinusoid fitted to the s4 coefficients derived from observations. The difference could partly originate from errors in the distortion observed, but it is likely to be due to the direct effects of the centrifugal force and the magnetic field.

Observations from RHESSI (Fivian et al. 2008) shows in the data obtained since 2004, an excess of oblateness of around 2 mas that the authors attribute to the EUV limb brightness. This excess could be merely the increase of oblateness due to the differential rotation. In other words, the excess could be explained by the combined contribution of the oblateness (quadrupolar term) and the hexadecapolar term.

It results from this discussion that the solar shape is nearly purely oblate near solar maximum but has a significant hexadecapole shape near minimum.

Exploring the Temporal Variation of the Solar Gravitational Moments

Various authors have independently developed high precision of Lunar and Planetary Ephemerides, which has served as a basis to set up celestial and nautical almanacs. They are considered as a worldwide resource for fundamental astronomical data, often being the first publications to incorporate new International Astronomical Union resolutions. Planetary Ephemerides are developed on the basis of numerical integration of the motion of the nine planets and the Moon fitted to the most accurate available observations. They progressively integrated the motion of perturbing main belt asteroids (up to 300), the Earth's rotation and Moon libration. These datasets comprise of a variety of planetary observations, such as, spacecraft radio science and tracking data (VLBI), direct radar measurements, lunar laser ranging (LLR), astrometric photographs, CCD and occultations to name a few. J2 is obtained as a sub-product of the fitting process assuming a complete consistency of the dynamical modeling since Earth rotation, Moon libration and asteroid orbits are integrated with the main equations of the planetary motions (see for instance a discussion by Hilton and Hohenkerk, 2010). The J2 estimate has thus been regularly published as shown in Table 1.

Table 1: Estimates of the solar quadrupole moment J2, with their uncertainties, deduced from the Ephemerides. JPL: NASA Jet Propulsion Laboratory in Pasadena (USA). IAA: Institute of Applied Astronomy in Saint Petersburg (Russia). IMCCE: Institut de M´ecanique C´eleste et de Calcul des Eph´em´erides in Paris (F). INPOP13c is an upgraded version of INPOP13a, fitted to LLR (Laser Lunar Ranging) observations, including new observations of Mars and Venus deduced from MEX, Mars Odyssey and VEX tracking data.

Name of the ephemeris & Institute	Quadrupole moment (in 10^{-7})	Author & Reference
DE405 JPL (USA)	1.9 ± 0.3	Standish 1998
EPM2004 IAA (Russia)	1.9 ± 0.3	Pitjeva 2005; Pitjeva 2013

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INPOP06 IMCCE (F)	1.95 ± 0.55	Fienga 2011
EPM2008 IAA (Russia)	1.92 ± 0.30	Pitjeva 2014a
INPOP08 IMCCE (F)	1.82 ± 0.47	Fienga 2011
DE423 JPL (USA)	1.80	Folkner 2015
INPOP10e IMCCE (F)	1.8 ± 0.25	Verma 2014
EPM2011 IAA (Russia)	2.0 ± 0.2	Pitjeva 2014a
DE430 JPL (USA)	2.1 ± 0.7	Folkner 2015
EPM2013 IAA (Russia)	2.22 ± 0.23	Pitjeva 2014a
INPOP13a IMCCE (F)	2.40 ± 0.20	Verma 2014
INPOP13c IMCCE (F)	2.3 ± 0.25	Fienga 2015
MESSENGER JPL (USA)	2.26 ± 0.09	Park 2017

A careful analysis of this set of data leads to a possible temporal dependance of J2 as shown in Fig. 5, redrawn from Fig.1 taken in Eren & Rozelot (2020).

A rather complete approach was first made by Komm et al. (2003), who derived from helioseismic data a temporal variation of the coefficient of P4 (in the Legendre expansion of the rotation law), indicative of torsional oscillation patterns extending deep into the convection zone, while the angular momentum shows a 1.3–year period that hints at a long-term trend likely related to the solar activity cycle. Antia et al. (2004) found also through an analysis of the helioseismic rotational splitting data a significant temporal variation in the angular momentum and in the gravitational multipole moments. For these authors, the quadrupole moment J2 exhibits no noticeable temporal variation, while J4 up to J12 do show a stronger variability.

To interpret the discrepancy with the Armstrong and Kuhn (1999) results, Antia et al. (2008) argue that “most of the distortion from sphericity is the direct response to the centrifugal force on the rotating surface layers, and not from the asphericity of the gravitational field”. This argument is in contradiction with Dicke’s statement: “a solar ellipticity requires a rotating distorted gravitational potential function, the most likely being a fast-rotating solar core. There are only two feasible sources of internal stress to distort the solar core, a magnetic field and a Reynolds stress (fluid motion). The latter does not readily yield a rotating distortion, but a magnetic field demands it”. Investigations of the properties of the solar interior provide additional indirect support, both experimental and theoretical, for the hypothesis that the Sun may contained a decoupled rapidly rotating core. Such a premise seems confirmed by Fossat et al. (2017) who analyzed the very low frequency g modes in the asymptotic regime by statistical means of the GOLF measurements on-board the SOHO satellite (Gabriel et al. 1995). They showed a rapid rotation frequency of 1644 ± 23 nHz of the solar core, which is a factor of 3.8 ± 0.1 faster than the rotation of the radiative envelope.

If such a fast solar rotator is confirmed, for a rigid rotation, this would imply a high solar core quadrupole moment (of about 3.6 x 10−5), in any case greater than the one which has been observed up to now (at the surface but integrated from the core). The question which arises, and presently into consideration, is how to take into account such an inside quadrupole moment with an integrated one of lower estimate. Two possibilities occur: one is through the Brans-Dicke theory, for which it would be conceivable to determine the υ coupling factor; the other one would be through an internal magnetic field compatible with the axial symmetry required by a field locked in the rotating core. Such a rapid rotation has not been confirmed and remains difficult to explain by models describing a pure angular momentum evolution without adding new dynamical processes such as internal magnetic breaking, which could have appeared when the Sun was young (Turck-Chi`eze et al. 2010). Such an interpretation is requiring further theoretical work which is also in progress.

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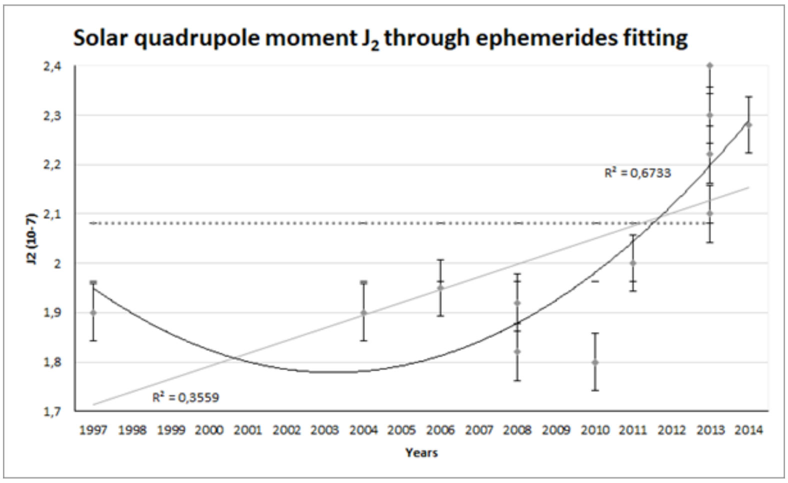

Figure 5: Solar quadrupole moment J2 estimates obtained as a sub-product of analytical solutions for the motions of planets deduced from high precision Lunar and Planetary Ephemerides. The best fit is a quadratic curve, but to be confirmed, would require a longer time scale. The main interest lies in the visible trend which is far from the (horizontal) dotted line which shows the weighted mean estimate over the ranging time.

Analysis of the perihelion precession of planetary orbits computed in the solar equatorial coordinate system, instead of the ecliptic coordinate system usually used, shows that a periodic variation of the J2 term rather than of a simple constant, must be considered, and can be described as followed (Xu et al., 2017).

According to its time dependence, the J2 term is redefined as J2 = J20 + J21, where J20 remains as the constant part of J2, while J21 denotes the temporal variation of J2, and can be expressed as

$$J_{21}=A\cos(\omega_{J_{21}} t)+B\sin(\omega_{J_{21}} t)=\sqrt{A^2+B^2}\,\sin(\omega_{J_{21}} t+\varphi)$$

The disturbing part of the heliopotential of Eq. 5 can be written as

$$\Phi=-\frac{GR_{eq}^2}{2r^3} J_2 (\sin^2\theta-1)$$

so that the secular effects of the solutions obtained by the integration for the Lagrangian equations of planetary motion can be developed according to J21.

Results are given in Xu et al. (2017) and the perihelion precession of Mercury's orbit caused by the periodic variation of J2 is given in Fig. 6. It can be seen that the maximum perihelion precession of Mercurys orbit caused by a periodic variation of J2 (of one solar cycle of 11.4 year, as suggested by Dicke, Kuhn & Libbrecht (1986) or Rozelot, Damiani & Pireaux (2009b)) is about 1700 J20, which is nearly 0.8 per cent of the secular perihelion precession.

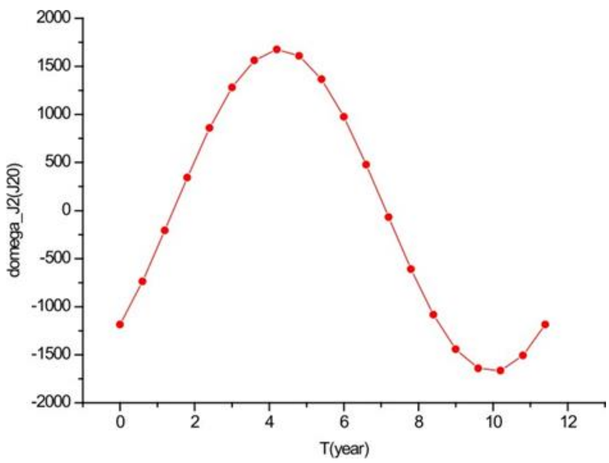

Figure 6: The perihelion precession of Mercury's orbit caused by a periodic variation of J2, here of one solar cycle.

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Relevance of accurate solar gravitational moments

Solar Astrophysical Quantities

A number of questions are still pending about solar global properties. Among them: what is the influence of solar core dynamics on the values of the global solar spin [J ⃗_⊙], on the multipolar moments [Jn] and on the solar shape coefficients [sn] (which reflect the internal non-homogeneous mass distribution and non-uniform angular velocity)? How to reconcile different estimates of Jn? How may the temporal dependence of Jn, cn, and J ⃗_⊙ with the solar cycle affect the planets' motion?

All these parameters are at the crossroad of solar physics, astrometry, and celestial mechanics. From the point of view of solar physics, their values reflect the physics of solar models: non-rigid rotation, solar latitudinal rotation, solar core properties, solar-cycle variations, and stellar evolution. The knowledge of their temporal variations is essential to set constraints on solar-cycle modeling or solar-evolution theories. Another issue addresses the response of the rotating body: why it does include higher multipole moments than the first one which is represented by the rotational potential [(1/2)ω2sin2(θ)]? Hubbard (1975) postulated that the "higher moments would be zero if there were no contribution from the most outer shell, or, in other words, that the quadrupole response to a quadrupole perturbation does not produce a purely quadrupole structure in the shell, which, in turn excites the higher multipole moments".

In any case, the anomalous precession of the orbit of Mercury and other planets, primarily due to General Relativity (GR) effects, could be reconsidered through corrections due either to the presence of the Telisto/Planet Nine or that of a rapid solar core. More stringent tests rely on measuring the orbital precessions with much greater precision.

Relativistic Astrometry

Space-time is shaped by the presence of solar system bodies. The space–time curvature induced by the Sun leads to light deflection or corresponding time delays in the propagation of signals. Precise astrometry in the solar neighborhood will thus require precise knowledge of the solar quadrupole moment and spin. Indeed, in addition to the solar mass monopole contribution (≈ 1.75 arcsec light deflection at grazing incidence), there is a quadrupolar one (≈ 0.4 - 0.3 μarcsec at grazing incidence) and rotational solar contribution (≈ ± 0.7 μarcsec at grazing incidence) (Pireaux 2002). Unfortunately, the contribution of the J2-term to light deflection drops dramatically as the angle of incidence (i.e. the closest approach to the Sun) increases (non-grazing incidence). Hence arises the need for determining an accurate J2.

Relativistic Celestial Mechanics

The solar quadrupole moment also plays a role in celestial mechanics. The advance of the relativistic precession of the perihelion of planets is a known phenomenon (Pireaux and Rozelot, 2003), ≈ 43 arcsec per century in the case of Mercury (exactly 〖42〗^''9794). As planetary ephemerides are today fitted to observational data sets (tracking data of spacecrafts, VLBI angular positions, or flyby normal points), a precise knowledge of J2 and its variation are necessary to improve the fitting process. The statistics of the obtained postfit residuals may help to better constrain ephemerides. As an example, the knowledge of Mercury's orbit has made significant progress using this method, reaching a few meters to be compared with 800 m obtained through direct radar ranging (Cavanaugh et al., 2007).

Furthermore, the gravitational field associated with the mass quadrupole of the Sun creates a Newtonian perturbation on the orbit of the planets that has an effect on planetary spins and on the ecliptic plane. Up to now, the terms that contain the contribution to J2 implicitly assume that the Sun's equatorial plane and the orbital plane of the planet in question coincide, which is an approximation. Here also, progress must be undertaken even if some work has been done in this direction (Iorio, 2011; Xu, 2011). Finally, through solar system spin–orbit coupling, J2 and J ⃗_⊙will indirectly influence the orbital parameters of solar system bodies. For example, the Moon–Earth spin–orbit coupling propagates the influence of the solar quadrupole moment to the Moon. This has allowed, as soon as 1997, to set a dynamic upper limit to the solar quadrupole moment of J2 to be ≤ 3 x10−6 (R¨osch and Rozelot, 1996, Rozelot and R¨osch, 1997; Rozelot and Bois, 1997; Bois and Girard, 1999), through observed lunar librations, but a more improved approach remains to be made.

Possible Alternative Theories of Gravitation?

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Presently, there is still a strong correlation between post-Newtonian parameter [β] and J2 in planetary ephemerides. Hence, one cannot fit simultaneously for those two parameters, but a better knowledge of J2 would thus help long-term Solar system modeling.

Indeed, together with [γ], which encodes the amount of curvature of space-time per unit rest-mass, the post-Newtonian parameter [β] contributes to the relativistic precession of planets. The latter parameter encodes the amount of non-linearity in the superposition law of gravitation (with $\beta \equiv 1$ in General Relativity (GR)). Using a reasonable value for J2 and present best constraints on post-Newtonian parameters γ and β, GR is still in the contention, but there is room for alternative theories too (Pireaux and Rozelot, 2003). Today, the best values available for β and γ are

(γ-1) 105 = 2.1 ± 2.3 (Cassini mission, Bertotti et al., 2003)

(β-1) 104 = 1.2 ± 1.1 (Lunar Laser Ranging, LLR: Williams et al., 2009,

Equation 27)

and η x 104 = (4β - γ - 3) = 4.4 ± 4.5, through LLR also (Williams et al., 2009, Equation 26).

By accurately measuring the perihelion advances of several planets, it will be possible in the near future to decorrelate all of these quantities. Such a test is the purpose of several space missions, such as Beppi–Columbo, Gaia and Lator. This last mission has for its primary objective the measurement of the key post-Newtonian Eddington parameter γ with an accuracy of one part in 109 (a factor 30 000 beyond the present best result obtained by means of Cassini (Bertotti et al., 2003). Direct measurement of the solar quadrupole moment [J2] to an accuracy of one part in 200 of its size of around 10−7 is expected (Turyshev et al., 2009).

Currently, a first analysis of the Messenger flybys (Iorio 2011) shows that the Lense–Thirring precession in the case of Mercury may have been canceled to a certain extent by the competing precession caused by a small mismodeling in the quadrupole mass moment [J2] of the Sun, which could be estimated at 1.8×10−7, thus more compatible with RHESSI results:

J2 = (1.46 ± 1.0) ×10−7 (Fivian et al., 2008).

Accurate determination of the perihelion advance of a planet

The first time that the solar quadrupole moment was associated with the gravitational motion of Mercury was in 1895, when Newcomb (1895 attempted to account for the anomalous perihelion advance of this planet with a modified gravitational field manifested by an oblateness f of the Sun (the difference between the equatorial and polar radius Δr, reported to the equatorial one). Indeed, a few years before in 1859, Le Verrier had observed a deviation of Mercury's orbit from Newtonian's predictions, that could not be due to the presence of known planets. But the difference between the equatorial and polar diameters of the Sun of Δr = 500 mas (milli-arc-sec), as advocated by Newcomb, was soon ruled out by solar observations. And Einstein's new theory of gravitation, General Relativity, could account for almost all the observed perihelion advance. So, Mercury's perihelion advance readily became one of the cornerstones for testing General Relativity; even though, now, a contribution to the perihelion shift from the solar figure (though very less important than first suggested by Newcomb) cannot be discarded.

Keplerian laws of planet motions are solutions of the n-body gravitational problem. However, solar gravitational moments distort the gravitation force acting on the planets and disturb their Keplerian motions. Analytic solution of a planet orbit disturbed by the solar gravitational moments (solar oblateness, but also star oblateness in the case of exoplanets) can be derived. Except short -and long- periodic disturbances, secular disturbances lead to a perihelion precession and a nodal regression as well as a mean motion advancing. The magnitudes of the short-periodic perihelion precession could disturb the observation of the secular effect if the survey is made shorter than one Julian year. Transformation of the formulas from the solar equatorial plane to the ecliptic one has been discussed by Xu et al. (2011) who deduced the Mercury's, Venus'and Mars' secular perihelion precession as a function of J2. Numerical estimates in the following are in arcsec per Julian century (noted: as/Jc) taking J2 = 2x10−7 (Pireaux & Rozelot 2003, Pitjeva 2005):

• perihelion precession of Mercury's orbit: $\Delta\omega$ ⃗_1= 2.95694x105J2, which is 0.0591 (as/Jc),

• perihelion precession of Venus's orbit: $\Delta\omega$ ⃗_1= 6.28801x104 J2,

which is 0.01258 (as/Jc),

• perihelion precession of Mars's orbit: $\Delta\omega$ ⃗_1 = 2.95694x103 J2, which is 0.0013 (as/Jc).

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Such numerical values are in good agreements with published results, namely those of Iorio (2004), noticing that the observations for the perihelion precession must be done long-yearly to be not disturbed by the short-periodic effects. Inversely, the solar oblateness could be determined through observation of perihelion precession of a planet.

5. Conclusion

We emphasized here the need to accurately capture real solar latitudinal intensity limb variations over a long period of time, a still challenging task. We highlighted that limb's tiny departure from perfect circularity, i.e. the asphericity, is sensitive to the Sun's otherwise invisible interior conditions allowing us to learn empirically about flows and motions there that would otherwise only be guessed from theoretical considerations. We then recalled that the knowledge of precise solar gravitational moments [Jn] and their temporal dependence, can be used for the precise determination of solar parameters such as the angular momentum and its variation on the long term, that may help to locate the seat of the dynamo (Komm et al., 2003). The observed oblateness temporal variations, which remain very faint (a few mas in amplitude), lag the solar activity cycles by about 3 years (This point may be checked in future, as it has been observed that complex sunspot groups also lag the solar cycle about 2 years (Kilcik et al. 2011). A major change, which could be due to a possible exchange in magnitude between the two first solar moments (n = 2 and 4), occurred between solar cycles 23 and 24: the oblateness was greater in cycle 24 despite the lower solar activity level (Rozelot and Kosovichev, 2020).

Furthermore, J2 at first, and surely J4, are at the crossroads of relativistic astrometry and relativistic celestial mechanics. Indeed: (i) one of the major effects of a dynamical flattening of the Sun due to its oblique rotation with respect to the ecliptic plane is a secular variation in the orbital elements of the planet, which is not negligible, and must not be neglected in the computation of the ephemerides of celestial bodies. And (ii) they must be used, via their decorrelation with PPN parameters (Post Newtonian), as tests of alternative theories of gravitation.

Detailed analysis of the photospheric intensity distribution, including the brightness at the extreme limb, is important to determine the solar shape and its underlying parameters, shape coefficients, and oblateness, for which the most likely theoretical value of Δr is 7.77 mas at null activity (best observed values give 8.21 mas ±1 mas, enclosing the theoretical estimate). The accurate knowledge of all above mentioned quantities is certainly of broad importance in solar and stellar physics, mainly concerning solar–cycle modeling and time evolution. The possible brightness variations include not only the classical limb-darkening function but also several other capabilities that enable us to deal with both the surface and the interior structure as well. The analysis must completely eliminate possible contributions to the solar photospheric intensity from an abundance of unresolved magnetic elements of very low magnetic flux density. First results obtained so far in this way from space missions (such as SDO - Solar Disk Observatory- Scherrer et al., 2012) seem to be reached. Furthermore, associated with other space missions dedicated to test the curvature of the solar system's gravity field with an accuracy better than 1 part in 109, the determination of the gravitational moments will allow us to reach the solar core velocity rate with an unequaled precision.

At last, the solar gravitational moments knowledge can be applied to stars and their planets. The dynamical influence of stellar oblateness may be approximated using the leading order quadrupole terms, neglecting those of order (f2).

In the case of an exoplanet of mass mp orbiting around its host star of radius

r∗ and mass M∗, the disturbing part of the stellar potential is thus written as

D=(GM_* m_p)/(2a_p) (R_*/a_p)^2 J_2 (3/2 〖sin〗^2 i_p-1) (26)

where ap is the distance of the planet from its host and i its inclination orbit. As an example, the precession rates of planets orbiting the rapidly-rotating main- sequence stars WASP- 33, Kepler-13A and HAT-P-7 reveal associated values of J2,stars of the order of 104.

Annex to subsection

"Surfaces of equal potential and level surfaces.

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Classical mathematical formalism"

Motion for an ideal fluid. The starting point is to consider the equation

$$\rho \ddot{\vec{x}} = \rho \vec{g} - \vec{\nabla} p \quad (27)$$

where $\ddot{\vec{x}}$ is the acceleration of the fluid, ρ the density, $\vec{g}$, the force acting on particles, and $\vec{\nabla} p$, the pressure gradient. For hydrostatic equilibrium, there is no motion, hence $\ddot{\vec{x}}=0$. For a nonrotating body, the force acting on particles is given by $\vec{g}=-\vec{\nabla}\varphi_{grav}$,

where φ_{grav} is the gravitational potential. Hence Eq. 27 reduces to

$$0=-\rho\vec{\nabla}\varphi_{grav}-\vec{\nabla}p \quad (28)$$

The curl of this equation is $\vec{\nabla}\rho \times \vec{\nabla}\varphi=0$, so that the normals to surfaces of constant ρ and φ point in the same direction. As a consequence, these surfaces coincide. Using the perfect gas equation of state (with constant chemical composition), it can be shown that surfaces of constant ρ, φ, temperature T and pressure P all coincide. This is known as the Von Zeipel theorem: any internal source of distortion in the gravitational field at the surface will manifest itself as a change of shape in the solar surface layer. Thus, measuring the shape of the surface layers is equivalent to measuring surfaces of constant gravitational potential.

When the body is rotating, the force acting on particles is the gradient of the gravity potential ($\varphi_{total} = \varphi_{grav} + \varphi_{rotation}$); so we must add the centrifugal force, so that Eq. 28 becomes

$$\vec{\nabla}p=-\rho\vec{\nabla}\varphi_{grav}+\zeta(r,\theta)\vec{s} \quad (29)$$

where $\zeta(r, \theta)$ is the angular velocity and $\vec{s}$ a vector perpendicular to the rotation axis directed outwards.

The equation to a surface of a spheroid whose semi-axes are R_{eq} and

$R_{pol} = R_{eq} (1 - f)$ is

$$\frac{r^2 \theta \cos^2\theta}{R_{eq}^2 (1-f)^2}+\frac{r^2 \theta \sin^2\theta}{R_{eq}^2}=1 \quad (30)$$

where r is the radius vector and θ the colatitude measured from the axis of revolution. Expanding $r(\theta)$ in powers of f it comes

$$r(\theta)=R_{eq}\left[1-f\cos^2\theta-\frac{3}{2}f^2(\sin^2\theta\cos^2\theta)+\frac{1}{8}f^3(1-5\sin^2\theta)\sin^2 2\theta+\cdots\right] \quad (31)$$

to which corrections can be added to take into account the distorted shape.

Density and pressure surfaces.

As defined in Armstrong and Kuhn (1999), let ρ the density (function of the radius r) and denote with a subscript 0, the lowest order l, spherically symmetric.

Asphericities, described as

$$c_l=-\frac{\rho_l}{d\rho^0/dr}, \text{(density)}, \quad (32)$$

$$s_l=\frac{-p_l}{dp^{(0)}/dr}, \text{(pressure)} \quad (33)$$

measure the perturbation (nonspherically symmetric) and are usually expressed in terms of the normalized potential defined by $J_l=K\emptyset_l$, where $K=R_\odot/GM_\odot$ at the solar surface. The different gravitational moments can be written as

$$J_{2l}=\frac{R_\odot}{GM_\odot}\emptyset_{2l}(R_\odot) \quad (34)$$

where $\emptyset_{2l}=0$ at the surface $r=R_\odot$. The function $\emptyset_{2l}$ is the solution to a differential equation requiring the knowledge of $\rho(r)$ and $\omega(r, \theta)$, where θ is the colatitude. A complete expression of $\emptyset_2$ and $\emptyset_4$ was provided by Armstrong and Kuhn (1999), using the standard rotation law, which permits to deduce cl:

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$$\frac{d^2\Phi_2}{dr^2}+\frac{2}{r}\frac{d\Phi_2}{dr}-6\frac{\Phi_2}{r^2}=\frac{4\pi^2}{M_r}\left[\Phi_2\frac{d\rho_0}{dr}-\frac{8}{21}\frac{2\omega_2}{\omega_0}r\rho_0\omega_0^2-\frac{r^2}{3}\frac{d}{dr}\left(\rho_0\omega_0^2+\frac{3}{7}\rho_0\frac{2\omega_2}{\omega_0}\omega_0^2\right)\right], \quad (35)$$

$$\frac{d^2\Phi_4}{dr^2}+\frac{2}{r}\frac{d\Phi_4}{dr}-\frac{20}{r^2}\Phi_4=\frac{4\pi^2}{M_r}\left[\Phi_4\frac{d\rho_0}{dr}+\frac{4}{35}\frac{2\omega_2}{\omega_0}r\rho_0\omega_0^2-\frac{2}{35}r^2\frac{d}{dr}\left(\frac{2\omega_2}{\omega_0}\rho_0\omega_0^2\right)\right]. \quad (36)$$

Results are given in Armstrong and Kuhn (1999) showing that the observations are somewhat inconsistent with current rotation models. However, it could be noted that a subsurface shear layer results when the helioseismically obtained internal rotation is matched with the surface rotation. Such an approach, using helioseismic data will be made in a second paper.

References

Ajabshirizadeh, A., J.P. Rozelot, J.P., Fazel, Z., Contribution of the Solar mag- netic field on Gravitational Moments, Scientia Iranica, Vol. 15, No. 1, 144–149 (2008).

Antia, H.M., Chitre, S.M., Gough, D.O., Temporal variations in the Suns rota- tional kinetic energy. A & A, 477, 657–663 (2008).

Armstrong, J., Kuhn, J.R., Interpreting the solar limb shape distortions. Astro- phys. Jour. 525, 533 (1999).

Bertotti, B., Farinella, P., Vokrouhlicky, D., in “Physics of the Solar System. Dynamics and Evolution, Space Physics, and Spacetime Structure”, Springer, ed. (2003).

Bois, E., Girard, J.-F., Impact of the quadrupole moment of the Sun on the dynamics of the EarthMoon system, Cel. Mech. 73, 329–338 (1999).

Bruns, H., “Die Figur der Erde”, Pub. K¨onigl. Preuss. Geodid¨at Inst., Berlin (1878).

Cavanaugh, J.F., Smith, J.C., Sun, X., Bartels, A.E., Ramos-Izquierdo, L., Krebs, D.J., McGarry, J.F., Trunzo, R.M., Novo-Gradac, A., Britt, J.L., Karsh, J., Katz, R.B., Lukemire, A.T., Szymkiewicz, R., Berry, D.L., Swinski, J.P., Neumann, G.A., Zuber, M.T., Smith, D.E., Space Sci. Rev. 131, 451 (2007).

Chandrasekhar, S. & Lebovitz, W., On the oscillations and the stability of rotating gaseous masses., Astrophys. Jour., 136, 1069–1081 (1962).

Chandrasekhar, S., The Equilibrium and the Stability of the Riemann Ellipsoids. I, Astrophys. Jour., 142, 890—921 (1965).

Chandrasekhar, S., The Equilibrium and the Stability of the Riemann Ellipsoids. II, Astrophys. Jour., 145, 842–877 (1966).

Clairaut, A.C., “Thorie de la Figure de la Terre”, in Principes de l’Hydrostatique, Durand ed., Paris (1743).

Darwin, G.H., The theory of the figure of the Earth carried to the second order, Monthly Not. Roy. Astro. Soc., 60, 82–124, (1899).

Debras and Chabrier, A complete study of the precision of the concentric MacLaurin spheroid method to calculate Jupiter’s gravitational moments, A & A, 609, A97 (2018).

Dicke, R.H., The Oblateness of the Sun and Relativity, Science, 184, Issue 4135, 419–429 (1974).

Dicke, R.H., Kuhn, J.R., Libbrecht, K.G., The variable oblateness of the Sun- measurements of 1984, Astrophys. Jour. 311, 1025–1030 (1986).

Emilio, M., Bush, R.I., Kuhn, J., Sherrer, P., A changing solar shape, Astrophys. Jour., 660, L161–L163 (2007).

Ess´en, H., The Physics of Rotational Flattening and the Point Core Model, International Journal of Geosciences, 2014, 5, 555–570 (2014).

Fienga, A., Laskar, J., Kuchynka, P., Manche, H., Desvignes, G., Gastineau, M., Cognard, I., Theureau, G., The INPOP10a planetary ephemeris and its applications in fundamental physics. Celest. Mech. Dyn. Astr. 111, 363–385 (2011).

Fienga, A., Laskar, J., Exertier, P., Manche, H., Gastineau, M., Numerical esti- mation of the sensitivity of INPOP planetary ephemerides to general relativity parameters. Celest. Mech. Dyn. Astr. 123, 325–349, (2015).

DOI:10.31401/SunGeo.2022.01.03

Fivian, M.D., Hudson, H.S., Lin, R.P., Zahid, H.J., Solar shape measurements from RHESSI: a large excess oblateness, Science 322 (5901), 560–562 (2008).

Folkner, W.M., Williams, J.G., Boggs, D.H., Park, R.S., Kuchynka, P., IPN Progress Report 42-196 February 15, 2014. The Planetary and Lunar Ephemerides DE430 and DE431 (2014).

Fossat, E., Boumier, P., et al., Asymptotic g modes: Evidence for a rapid rotation of the solar core, A & A, 604, 17 (2017).

Godier, S., Rozelot, J.P., A new outlook on the 'Differential Theory' of the solar Quadrupole Moment and Oblateness, Solar Phys., 199, 217–229 (2001).

Hilton, J.L., Hohenkerk, C.Y., A comparison of the high accuracy planetary ephemerides DE421, EPM2008, and INPOP08. Proceedings of the Journ´ees 2010 Syst`emes de r´ef´erence spatio-temporels (JSR2010): New challenges for reference systems and numerical standards in astronomy, Observatoire de Paris, 2022 September 2010, Edited by Nicole Capitaine, p. 77–80 (2010).

Howard, R., Gilman, P. A., Gilman, P. I., Rotation of the Sun measured from Mount Wilson white-light images, Astrophys. Jour., 283, 373–384 (1984).

Hubbard, W.B., Gravitational field of a rotating planet with a polytropic index of unity, Sov. Astron. Vol. 18, No5 621–625 (1975).

Hubbard, W.B., Effects of differential rotation on the gravitational figures of Jupiter and Saturn, Icarus, 52, 509–515 (1982).

Hubbard, W.B., High-precision Maclaurin–based models of rotating liquid planets, Astrophys. Jour. L., 756, L15 (2012).

Iorio, L., Mercury and frame-dragging in light of the MESSENGER flybys: conflict with general relativity, poor knowledge of the physical properties of the Sun, data reduction artifact, or still insufficient observations? arXiv:1109.0266 (2011).

Isaak, G.R., Solar core rotation, in European Meeting on Solar Physics, 12-18 September 1999, Florence (I), ESA SP–448 (1999).

Jacobi, C. G. J., De usu legitimo formulae summatoriae Maclaurinianae, Journal fu¨r die Reine und Angewandte Mathematik, Volume 1834: Issue 12, 263-281 (1834).

James, R. A., The structure and stability of rotating gas masses, Astrophys. Jour., 140, 552–582 (1964).

Javaraiah, J. and Komm, R.W.: "Sun's Rotation", Nova Science Publishers, Inc., Javaraiah, J. and Gokhale, M.H. eds. (2002).

Javaraiah, J., Rozelot, J.P., technical papers CERGA observatory (2002).

Jeans, J. H., Gravitational instability and the nebular hypothesis, Phil. Trans. Roy. Soc. London, 213, 457–485 (1914).

Klioner S. A., A Practical Relativistic Model for Microarcsecond Astrometry in Space, Astronom. Jour., 125, 1580–1597 (2003).

Kilcik, A., Yurchyshyn, V. B., Abramenko, V., Goode, P. R., O¨ zgu¨¸c, A., Rozelot,

J. P., Cao, W.,Time Distributions of Large and Small Sunspot Groups Over Four Solar Cycles, Astrophys. Jour., 731:30 (8pp), (2011).

Komm, R., Howe, R., Durney, B.R., Hill, F., Temporal Variation of Angu- lar Momentum in the Solar Convection Zone, Astrophys. Jour., 586, 650–662 (2003).

Kuiper, J., in "The Sun", Chicago University Press (USA) (1972).

Lefebvre, S., Kosovichev, A.K., Rozelot, J.P., Helioseismic Test of Nonhomolo- gous Solar Radius Changes with the 11 Year Activity Cycle, Astrophys. Jour., 658, L135–L138 (2007).

Lydon, T.J., Sofia, S., A Measurement of the Shape of the Solar Disk: The Solar Quadrupole Moment, the Solar Octopole Moment, and the Advance of Perihe- lion of the Planet Mercury, Phys. Rev. Lett. 76, 177–179 (1996).

Lyttleton, R.A., The Stability of Rotating Liquid Masses, Cambridge University Press (1953).

Maclaurin, C., in Thomson, W. & Tait, P. G., Treatise on Natural Philosophy, Cambridge University Press, Part II, Art. 772 (1742).

DOI:10.31401/SunGeo.2022.01.03

Maeder, A., Stellar evolution with rotation IV: von Zeipel's theorem and anisotropic losses of mass and angular momentum, A & A, 347, 185–193 (1999).

Milne, E.A., The equilibrium of a rotating star, Monthly Notices Royal Astron. Soc., 83, 119–147 (1923).

Modolenvsky, M.S., Geodezika i Kartografiya, 5, 11 (1988).

Moritz, H., "The Figure of the Earth", Wichmann ed. Paris (1980).

Newcomb, S., Constants of Astronomy, in: "The Elements of the Four Inner Planets and the Fundamental Constants of Astronomy" US GPO, Washington, DC (USA), p. 111 (1865).

Ostriker, J.P., Mark, J.W.K., Rapidly rotating stars. I. The self-consistent-field method, Astrophys. Jour., 151, 1075–1088 (1968).

Panhans M., Soffel M. H., Gravito–magnetism of an extended celestial body, Classical Quant. Grav., 31, 245012 (2014).

Park, R.S., Folkner, W.M., Konopliv, A.S., Williams, J.G., Smith, D.E., Zuber, M.T., Precession of Mercurys Perihelion from Ranging to the MESSENGER spacecraft. Astronom. Jour. 153 (3), 121 (2017).

Patern´o, L., Sofia, S., DiMauro, M.P., The rotation of the Suns core. A & A 314, 940–946 (1996).

Pijpers, F.P., Helioseismic determination of the solar gravitational quadrupole moment, Monthly Notices Roy. Soc. 297, L76–L80 (1998).

Pireaux, S., Signature d'observables en th´eories alternatives de la gravitation. Relativit´e G´en´erale et Cosmologie Quantique. Universit´e Catholique de Louvain, (2002).

Pireaux, S., Rozelot, J.P., Solar quadrupole moment and purely relativistic gravitation contributions to Mercury's perihelion advance, Astrophys. Space Sci. 284, 1159–1194 (2003).

Pireaux, S., Lefebvre, S., Rozelot, J.P., Solar gravitational moments and solar core dynamics, SF2A-2005: Semaine de l'Astrophysique Fran¸caise, meeting held in Strasbourg, France, June 27-July 1, 2005, Edited by F. Casoli, T. Contini, J.M. Hameury and L. Pagani. Published by EdP-Sciences, Conference Series, 2005, p. 121 (2005).

Pitjeva, E.V., Relativistic effects and solar oblateness from radar observations of planets and spacecraft, Astron. Lett. 31 (5), 340–349 (2005).

Pitjeva, E.V., Updated IAA RAS Planetary Ephemerides-EPM2011 and their use in scientic research. Solar System Research, Vol. 47, No. 5, pp. 386402. Original Russian Text published in Astronomicheskii Vestnik, 2013, Vol. 47, No. 5, pp. 419–435 (2013).

Pitjeva, E.V., Pitjev, N.P., Development of planetary ephemerides EPM and their applications, Celest. Mech. Dyn. Astr. 119 (3), 237–256 (2014).

Poincar´e, H., in "Tisserand, F., M´ecanique C´eleste", Gauthier-Villars ed., Paris, 2, chap. xi (1891).

Radau, R., "Remarques sur la Th´eorie de la Figure de la Terre", Bull. Astron., 2, 157 (1885).

Riemann, B., Abb. d. K¨onigl. Gessel, der Wis. d. Gottingen, 9, 3–36 (1860).

Rieutord, M., Espinosa Lara, F., & Putigny, B. J., Problems raised by (fast) rotating stars, Comput. Phys., 318, 277 (2016).

Rozelot and R¨osch, An Upper Bound to the Solar Oblateness, Sol. Phys., 11–18 (1997).

R¨osch, J., Rozelot, J.P., Le Soleil change-t-il de forme?, Comptes Rendus Acad. Sci. Paris, s´erie II, 322 637–644 (1996).

Rozelot, J.P., Lefebvre, S., Desnoux, V., Solar limb shape distortions, Solar Phys. 217, 3952 (2003).

Rozelot, J.P., Bois, E., New results concerning the solar oblateness and conse- quences on the solar interior. In: Balasubramaniam, K.S., Rabin, D.M. (Eds.), 18th NSO Workshop, Sacramento Peak, USA, Conference of the Pacific Astro- nomical Society, no. 140, pp. 75–82 (1997).

Rozelot, J.P., Lefebvre, S., The Figure of the Sun, astrophysical consequences. In: ''The Sun's Surface and subsurface, Investigating Shape and Irradiance", Lecture Notes in Physics, vol. 599, Springer, Heidelberg (D), 4–27 (2003).

Rozelot, J.P., Damiani, C., Pireaux, S., Probing the solar surface: the oblateness and astrophysical consequences, Astrophys. Jour. 703, 1791–1796 (2009).

DOI:10.31401/SunGeo.2022.01.03

Rozelot, J.P., Damiani, C., Lefebvre, S., Variability of the solar shape (before space dedicated missions), Jour. Atmos. Sol. Terr. Phys, 71, 1683-1694 (2009).

Rozelot, J.P., Fazel, Z., Revisiting the Solar Oblateness: Is Relevant Astro- physics Possible? Sol. Phys., 287, 161170 (2013).

Rozelot, J.P., Eren, S., Exploring the temporal variation of the solar quadrupole moment from relativistic gravitation contributions: A fortuitous circumstance? Ad. Sp. Research, 65, 2821–2827 (2020)

Rozelot, J.P., Kosovichev, A.K., Kilcik, A., Solar oblateness & asphericities temporal variations: outstanding some unsolved issues, in "Solar and Stellar Magnetic Fields: Origins and Manifestations", IUA symposium 354, A.G. Kosovichev, K. Strassmeier & M. Jardine, eds., 232–237 (2020).

Rumiantsev, V.V., The stability of Maclaurin ellipsoids of a rotating fluid, Jour- nal of Applied Mathematics and Mechanics, Vol.23, No.3, 494–504, (1959).

Scheiner, C., "Rosa Ursina sive sol ex admirando facularum (...)" (1630).

Scherrer, P. H., Bogart, R. S., Bush, R. I., Hoeksema, J. T., Kosovichev, A. G.,

Schou, J., Rosenberg, W., Springer, L., Tarbell, T. D., Title, A., Wolfson, C. J., Zayer, I., MDI Engineering Team, The solar oscillations investigation - Michelson Doppler Imager, Solar Phys., 162, 129–188 (1995).

Spalding, C., Millholland, S.C., Stellar Oblateness versus Distant Giants in Exciting Kepler Planet Mutual Inclinations, Astronomical Journal, 160, Issue 3, id.105 (2020).

Standish, M., JPL Interoffice Memoradum, IOM 312.F - 98-048, August 26, 1998, 18pp. (1998).

Stoeckly, R., Polytropic Models with Fast, Non-Uniform Rotation, Astrophys. Jour., 142, 208–228 (1965).

Turck-Chi`eze, Palacios, A., Marques, J. P., Nghiem, P. A. P.S., Seismic and Dy- namical Solar Models. I. The Impact of the Solar Rotation History on Neutrinos and Seismic Indicators, Astrophys. Jour., 715, 1539–1555 (2010).

Verma, A.K., Fienga, A., Laskar, J., Manche, H., Gastineau, M., Use of MES- SENGER radioscience data to improve planetary Ephemeris and to test general relativity. Astron. Astrophys. 561, A115 (2014).

Wavre, R., "Figures plan´etaires et g´eod´esie", Gauthiers–Villars ed., Paris (1932).

Williams J. G., Turyshev, S. G., Slava, G., Boggs, Lunar Laser Ranging Tests of the Equivalence Principle with the Earth and Moon, D. H., Int. J. Mod. Phys. D, 18, 1129–1175 (2009).

Xu, Y., Shen, Y., Xu, G., Shan, X., Rozelot, J.P., Perihelion precession caused by solar oblateness variation in equatorial and ecliptic coordinate systems, Monthly Notices Roy. Soc., 472, 2686–2693 (2017).

Xu, Y., Xu Y., Yang Y., Zhang Q., Xu G., Solar oblateness and Mercury's perihelion precession, Monthly Notices Roy. Soc., 415, 3335–33 (2011).

Yuan, D-N., Sjogren, W.L., Konopliv, A.S., Kucinskas, A.B., J. Geophys. Res. E, 106, 23377 (2002).

Zharkov, V.N. & Trubitsyn, Physics of Planetary Interiors, Pachart ed., Arizona Press (USA) (1978).

DOI:10.31401/SunGeo.2022.01.03